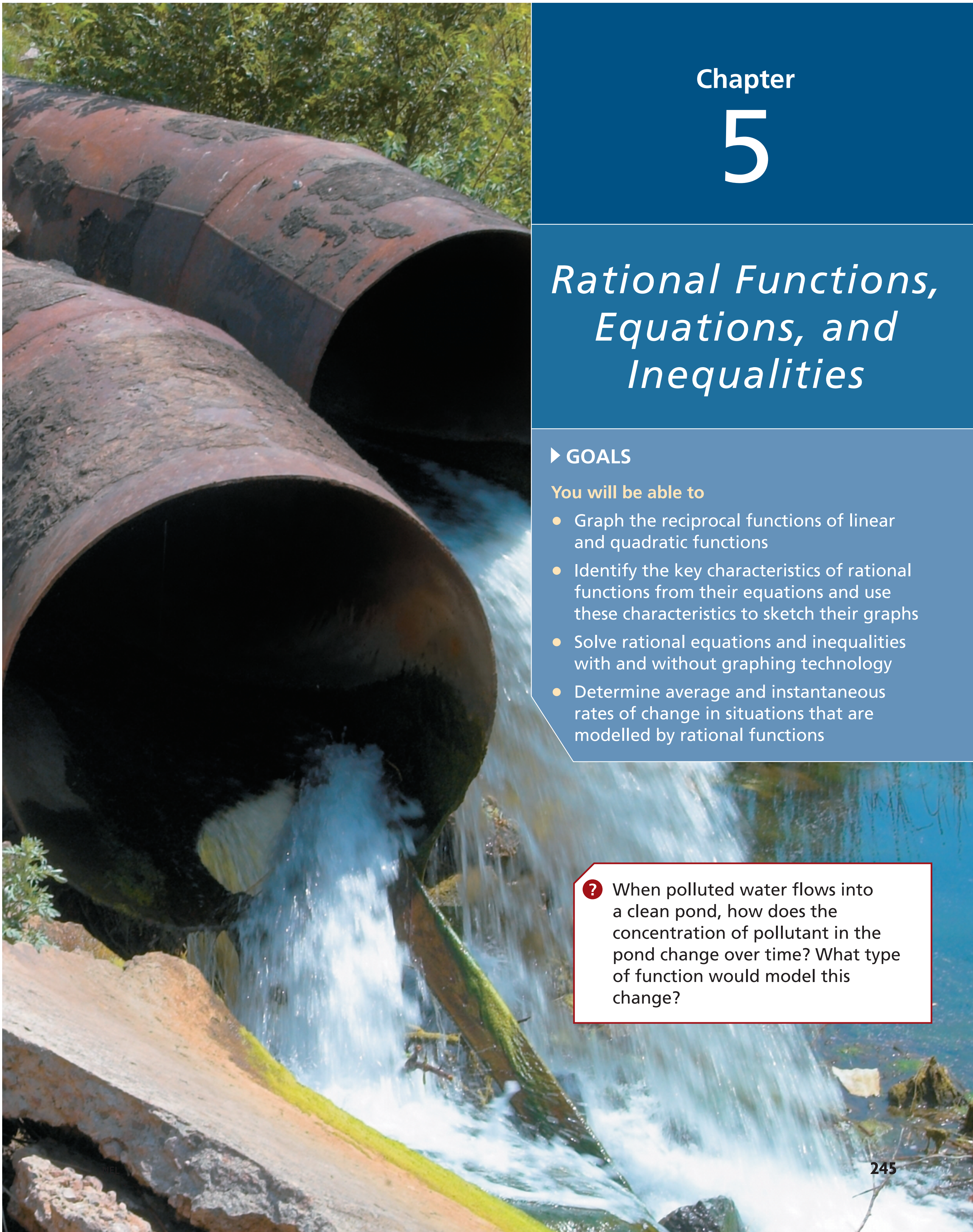


Chapter 5

Rational Functions, Equations, and Inequalities

GOALS

You will be able to

- Graph the reciprocal functions of linear and quadratic functions
- Identify the key characteristics of rational functions from their equations and use these characteristics to sketch their graphs
- Solve rational equations and inequalities with and without graphing technology
- Determine average and instantaneous rates of change in situations that are modelled by rational functions

? When polluted water flows into a clean pond, how does the concentration of pollutant in the pond change over time? What type of function would model this change?

5 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
1	R-3
2, 3, 4, 8	R-4
7	R-8

SKILLS AND CONCEPTS You Need

1. Factor each expression.

a) $x^2 - 3x - 10$

b) $3x^2 + 12x - 15$

c) $16x^2 - 49$

d) $9x^2 - 12x + 4$

e) $3a^2 + a - 30$

f) $6x^2 - 5xy - 21y^2$

2. Simplify each expression. State any restrictions on the variables, if necessary.

a) $\dfrac{12 - 8s}{4}$

b) $\dfrac{6m^2n^4}{18m^3n}$

c) $\dfrac{9x^3 - 12x^2 - 3x}{3x}$

d) $\dfrac{25x - 10}{5(5x - 2)^2}$

e) $\dfrac{x^2 + 3x - 18}{9 - x^2}$

f) $\dfrac{a^2 + 4ab - 5b^2}{2a^2 + 7ab - 15b^2}$

3. Simplify each expression, and state any restrictions on the variable.

a) $\dfrac{3}{5} \times \dfrac{7}{9}$

b) $\dfrac{2x}{5} \div \dfrac{x^2}{15}$

c) $\dfrac{x^2 - 4}{x - 3} \div \dfrac{x + 2}{12 - 4x}$

d) $\dfrac{x^3 + 4x^2}{x^2 - 1} \times \dfrac{x^2 - 5x + 6}{x^2 - 3x}$

4. Simplify each expression, and state any restrictions on the variable.

a) $\dfrac{2}{3} + \dfrac{6}{7}$

b) $\dfrac{3x}{4} + \dfrac{5x}{6}$

c) $\dfrac{1}{x} + \dfrac{4}{x^2}$

d) $\dfrac{5}{x - 3} - \dfrac{2}{x}$

e) $\dfrac{2}{x - 5} + \dfrac{y}{x^2 - 25}$

f) $\dfrac{6}{a^2 - 9a + 20} - \dfrac{8}{a^2 - 2a - 15}$

5. Solve and check.

a) $\dfrac{5x}{8} = \dfrac{15}{4}$

b) $\dfrac{x}{4} + \dfrac{1}{3} = \dfrac{5}{6}$

c) $\dfrac{4x}{5} - \dfrac{3x}{10} = \dfrac{3}{2}$

d) $\dfrac{x + 1}{2} - \dfrac{2x - 1}{3} = -1$

6. Sketch the graph of the reciprocal function $f(x) = \frac{1}{x}$ and describe its characteristics. Include the domain and range, as well as the equations of the asymptotes.

7. List the transformations that need to be applied to $y = \frac{1}{x}$ to graph each of the following reciprocal functions. Then sketch the graph.

a) $f(x) = \dfrac{1}{x + 3}$

b) $f(x) = \dfrac{2}{x - 1}$

c) $f(x) = -\dfrac{1}{2x} - 3$

d) $f(x) = \dfrac{2}{-3(x - 2)} + 1$

8. Describe the steps that are required to divide two rational expressions. Use your description to simplify $\dfrac{9y^2 - 4}{4y - 12} \div \dfrac{9y^2 + 12 + 4}{18 - 6y}$.

APPLYING *What You Know*

Painting Houses

Tony can paint the exterior of a house in six working days. Rebecca takes nine days to complete the same painting job.

? How long will Rebecca and Tony take to paint a similar house, if they work together?

A. What fraction of the job can Tony complete in one day? What fraction of the job can Rebecca complete?

B. Write a numerical expression to represent the fraction of the job that Rebecca and Tony can complete in one day, if they work together.

C. Let x represent the number of days that Rebecca and Tony, working together, will take to complete the job. Explain why $\frac{1}{x}$ represents the fraction of the job Rebecca and Tony will complete in one day when they work together.

D. Use your answers for parts B and C to write an equation. Determine the **lowest common denominator** for the **rational expressions** in your equation. Rewrite the equation using the lowest common denominator.

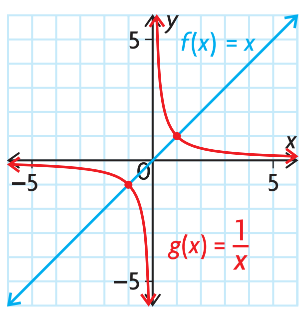
E. Solve the equation you wrote in part D by collecting like terms and comparing the numerators on the two sides of the equation.

F. What is the amount of time Rebecca and Tony will take to paint a similar house, when they work together?

5.1 Graphs of Reciprocal Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software



GOAL

Sketch the graphs of reciprocals of linear and quadratic functions.

INVESTIGATE the Math

Owen has noted some connections between the graphs of $f(x) = x$ and its reciprocal function $g(x) = \frac{1}{x}$.

- Both graphs are in the same quadrants for the same x-values.
- When $f(x) = 0$, there is a vertical asymptote for $g(x)$.
- $f(x)$ is always increasing, and $g(x)$ is always decreasing.

? How are the graphs of a function and its reciprocal function related?

A. Explain why the graphs of $f(x) = x$ and $g(x) = \frac{1}{x}$ are in the same quadrants over the same intervals. Does this relationship hold for $m(x) = -x$ and $n(x) = -\frac{1}{x}$? Does this relationship hold for any function and its reciprocal function? Explain.

B. What graphical characteristic in the reciprocal function do the zeros of the original function correspond to? Explain.

C. Explain why the reciprocal function $g(x) = \frac{1}{x}$ is decreasing when $f(x) = x$ is increasing. Does this relationship hold for $n(x) = -\frac{1}{x}$ and $m(x) = -x$? Explain how the increasing and decreasing intervals of a function and its reciprocal are related.

D. What are the y-coordinates of the points where $f(x)$ and $g(x)$ intersect? Will the points of intersection for any function and its reciprocal always have the same y-coordinates? Explain.

E. Explain why the graph of $g(x)$ has a horizontal asymptote. What is the equation of this asymptote? Will all reciprocal functions have the same horizontal asymptote? Explain.

F. On graph paper, draw the graph of $p(x) = x^2 - 4$. In a table like the one below, note the characteristics of the graph of $p(x)$ and use this information to help you determine the characteristics of the reciprocal function $q(x) = \frac{1}{x^2 - 4}$.

Characteristics	$p(x) = x^2 - 4$	$q(x) = \frac{1}{x^2 - 4}$
zeros and/or vertical asymptotes		
interval(s) on which the graph is above the *x*-axis (all values of the function are positive)		
interval(s) on which the graph is below the *x*-axis (all values of the function are negative)		
interval(s) on which the function is increasing		
interval(s) on which the function is decreasing		
point(s) where the *y*-value is 1		
point(s) where the *y*-value is -1		

G. On the same graph, draw the vertical asymptotes for the reciprocal function. Then use the rest of the information determined in part F to draw the graph for $q(x) = \frac{1}{x^2 - 4}$.

H. Verify your graphs by entering $p(x)$ and $q(x)$ in a graphing calculator using the "friendly" window setting shown.

I. Repeat parts F to H for the following pairs of functions.

a) $p(x) = x + 2$ and $q(x) = \frac{1}{x + 2}$

b) $p(x) = 2x - 3$ and $q(x) = \frac{1}{2x - 3}$

c) $p(x) = (x - 2)(x + 3)$ and $q(x) = \frac{1}{(x - 2)(x + 3)}$

d) $p(x) = (x - 1)^2$ and $q(x) = \frac{1}{(x - 1)^2}$

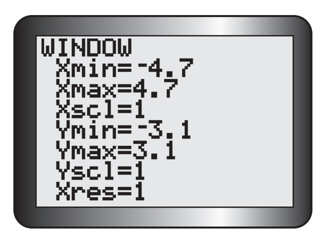
J. Write a summary of the relationships between the characteristics of the graphs of

a) a linear function and its reciprocal function

b) a quadratic function and its reciprocal function

Tech *Support*

On a graphing calculator, the length of the display screen contains 94 pixels, and the width contains 62 pixels. When the domain, *X*max-*X*min, is cleanly divisible by 94, and the range, *Y*max-*Y*min, is cleanly divisible by 62, the window is friendly. This means that you can trace without using "ugly" decimals. A friendly window is useful when working with rational functions.



Use brackets when entering reciprocal functions in the *Y* = editor of a graphing calculator. For example, to graph the function $f(x) = \frac{1}{x^2 - 4}$, enter $Y1 = \frac{1}{(x^2 - 4)}$.

Reflecting

K. How did knowing the positive/negative intervals and the increasing/decreasing intervals for $p(x) = x^2 - 4$ help you draw the graph for $p(x) = \frac{1}{x^2 - 4}$?

L. Why are some numbers in the domain of a function excluded from the domain of its reciprocal function? What graphical characteristic of the reciprocal function occurs at these values?

M. What common characteristics are shared by all reciprocals of linear and quadratic functions?

APPLY the Math

EXAMPLE 1 Connecting the characteristics of a linear function to its corresponding reciprocal function

Given the function $f(x) = 2 - x$,

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 2 - x$ is a linear function.

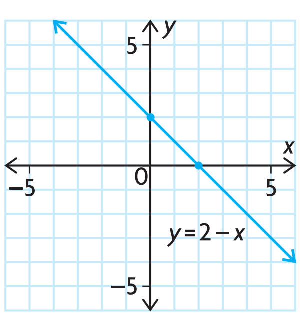
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R}\}$

The domain and range of most linear functions are the set of real numbers.

From the equation, the y-intercept is 2.
$f(x) = 0$ when $0 = 2 - x$
$x = 2$

A linear function $f(x) = mx + b$ has y-intercept b.
The x-intercept occurs where $f(x) = 0$.

The x-intercept is 2.



Sketch the graph of $f(x)$ to determine the positive and negative intervals. The line $y = 2 - x$ is above the x-axis for all x-values less than 2 and below the x-axis for all x-values greater than 2.

$f(x)$ is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.
$f(x)$ is decreasing when $x \in (-\infty, \infty)$.

This is a linear function with a negative slope, so it is decreasing over its entire domain.

b) The reciprocal function is $g(x) = \frac{1}{2 - x}$.

$D = \{x \in \mathbf{R} | x \neq 2\}$

$R = \{y \in \mathbf{R} | y \neq 0\}$

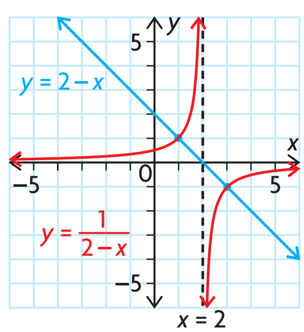
The y-intercept is 0.5.

The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 0$.

The reciprocal function is positive when $x \in (-\infty, 2)$ and negative when $x \in (2, \infty)$.

It is increasing when $x \in (-\infty, 2)$ and when $x \in (2, \infty)$.

The graph of $g(x) = \frac{1}{2 - x}$ intersects the graph of $g(x) = 2 - x$ at $(1, 1)$ and $(3, -1)$.



All the y-values of points on the reciprocal function are reciprocals of the y-values on the original function.

There is a vertical asymptote at the zero of the original function.
The reciprocals of a linear function always have the x-axis as a horizontal asymptote.

The positive/negative intervals are always the same for both functions.

Because the original function is always decreasing, the reciprocal function is always increasing.

The reciprocal of 1 is 1, and the reciprocal of -1 is -1. Thus, the two graphs intersect at any points with these y-values.

Use all this information to sketch the graph of the reciprocal function.

EXAMPLE 2 Connecting the characteristics of a quadratic function to its corresponding reciprocal function

Given the function $f(x) = 9 - x^2$

a) determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals

b) use your answers for part a) to sketch the graph of the reciprocal function

Solution

a) $f(x) = 9 - x^2$ is a quadratic function.

$D = \{x \in \mathbf{R}\}$

The domain of a quadratic function is the set of real numbers.

$f(0) = 9$, so the y-intercept is 9.

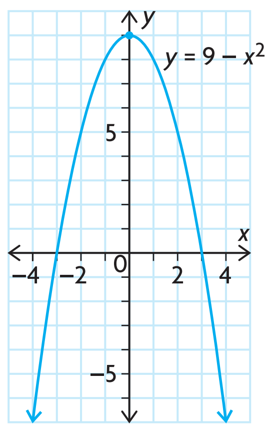
$R = \{y \in \mathbf{R} | y \leq 9\}$

The graph of $f(x)$ is a parabola that opens down. The vertex is at $(0, 9)$, so $y \leq 9$.

$$f(x) = 0 \Rightarrow 9 - x^2 = 0$$
$$(3 - x)(3 + x) = 0$$
$$x = \pm 3$$

Factor and determine the x-intercepts.

The x-intercepts are -3 and 3.



The parabola $y = 9 - x^2$ is above the x-axis for x-values between -3 and 3. The graph is below the x-axis for x-values less than -3 and for x-values greater than 3.

$f(x)$ is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$.

$f(x)$ is increasing when $x \in (-\infty, 0)$ and decreasing when $x \in (0, \infty)$.

The y-values increase as x increases from $-\infty$ to 0. The y-values decrease as x increases from 0 to ∞.

b) The reciprocal function is $g(x) = \frac{1}{9 - x^2}$.

The vertical asymptotes are $x = -3$ and $x = 3$.

$D = \{x \in \mathbf{R} \mid x \neq \pm 3\}$

Vertical asymptotes occur at each zero of the original function, so these numbers must be excluded from the domain.

The horizontal asymptote is $y = 0$.

The y-intercept is $\frac{1}{9}$.

The reciprocals of all quadratic functions have the x-axis as a horizontal asymptote. The y-intercept of the original function is 9, so the y-intercept of the reciprocal function is $\frac{1}{9}$.

There is a local minimum value at $\left(0, \frac{1}{9}\right)$.

$R = \left\{y \in \mathbf{R} \mid y < 0 \text{ or } y \geq \frac{1}{9}\right\}$

When the original function has a local maximum point, the reciprocal function has a corresponding local minimum point.

The reciprocal function is positive when $x \in (-3, 3)$ and negative when $x \in (-\infty, -3)$ and when $x \in (3, \infty)$. It is decreasing when $x \in (-\infty, -3)$ and when $x \in (-3, 0)$, and increasing when $x \in (0, 3)$ and when $x \in (3, \infty)$.

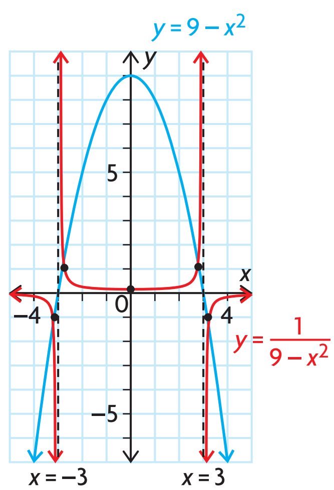
The positive/negative intervals are always the same for both functions. Where the original function is decreasing, excluding the zeros, the reciprocal function is increasing (and vice versa).

$$f(x) = 1 \text{ when } 9 - x^2 = 1 \qquad \text{and} \qquad f(x) = -1 \text{ when } 9 - x^2 = -1$$

$$\begin{aligned} -x^2 &= 1 - 9 \\ -x^2 &= -8 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \end{aligned} \qquad\qquad \begin{aligned} -x^2 &= -1 - 9 \\ -x^2 &= -10 \\ x^2 &= 10 \\ x &= \pm \sqrt{10} \end{aligned}$$

A function and its reciprocal intersect at points where $y = \pm 1$. Solve the corresponding equations to determine the x-coordinates of the points of intersection.

The graph of $g(x) = \frac{1}{9 - x^2}$ intersects the graph of $f(x) = 9 - x^2$ at $(-2\sqrt{2}, 1)$, $(2\sqrt{2}, 1)$ and at $(-\sqrt{10}, -1)$, $(\sqrt{10}, -1)$.



Use all this information to sketch the graph of the reciprocal function.

In Summary

Key Idea

- You can use key characteristics of the graph of a linear or quadratic function to graph the related reciprocal function.

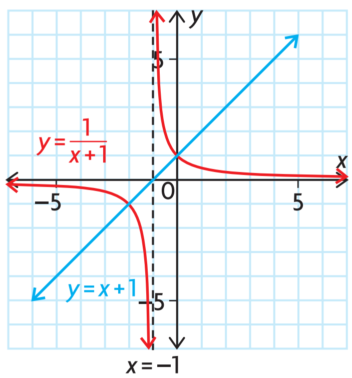
Need to Know

- All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
- The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- A reciprocal function will always have $y = 0$ as a horizontal asymptote if the original function is linear or quadratic.
- A reciprocal function has the same positive/negative intervals as the original function.

(*continued*)

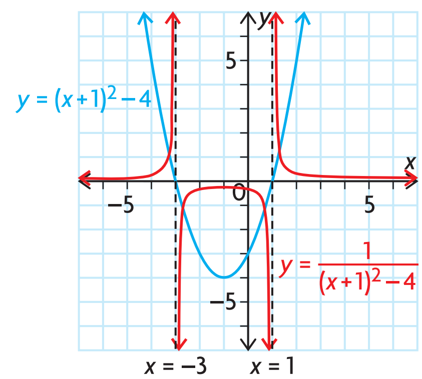
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1.
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

A linear function and its reciprocal



Both functions are negative when $x \in (-\infty, -1)$ and positive when $x \in (-1, \infty)$. The original function is increasing when $x \in (-\infty, \infty)$. The reciprocal function is decreasing when $x \in (-\infty, -1)$ or $(-1, \infty)$.

A quadratic function and its reciprocal

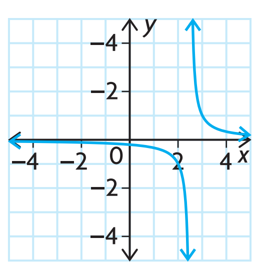


Both functions are negative when $x \in (-3, 1)$ and positive when $x \in (-\infty, -3)$ or $(1, \infty)$. The original function is decreasing when $x \in (-\infty, -1)$ and increasing when $x \in (-1, \infty)$. The reciprocal function is increasing when $x \in (-\infty, -3)$ or $(-3, -1)$ and decreasing when $x \in (-1, 1)$ or $(1, \infty)$.

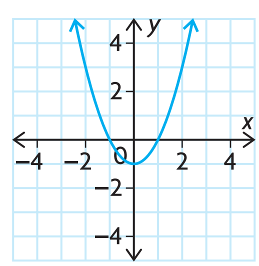
CHECK Your Understanding

1. Match each function with its equation on the next page. Then identify which function pairs are reciprocals.

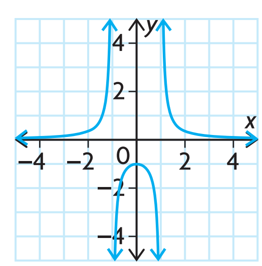
a)



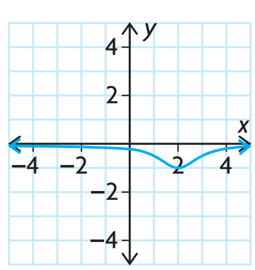
c)



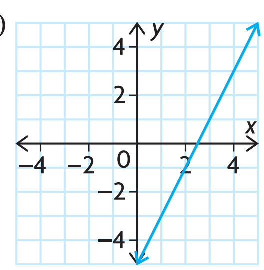
e)



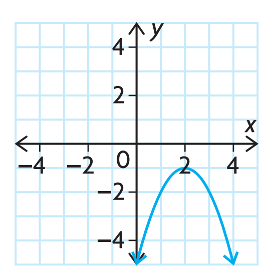
b)



d)



f)



A $y = \dfrac{1}{-(x-2)^2 - 1}$

B $y = \dfrac{1}{x^2 - 1}$

C $y = \dfrac{1}{2x - 5}$

D $y = x^2 - 1$

E $y = -(x-2)^2 - 1$

F $y = 2x - 5$

2. For each pair of functions, determine where the zeros of the original function occur and state the equations of the vertical asymptotes of the reciprocal function, if possible.

a) $f(x) = x - 6, g(x) = \dfrac{1}{x - 6}$

b) $f(x) = 3x + 4, g(x) = \dfrac{1}{3x + 4}$

c) $f(x) = x^2 - 2x - 15, g(x) = \dfrac{1}{x^2 - 2x - 15}$

d) $f(x) = 4x^2 - 25, g(x) = \dfrac{1}{4x^2 - 25}$

e) $f(x) = x^2 + 4, g(x) = \dfrac{1}{x^2 + 4}$

f) $f(x) = 2x^2 + 5x + 3, g(x) = \dfrac{1}{2x^2 + 5x + 3}$

3. Sketch the graph of each function. Use your graph to help you sketch the graph of the reciprocal function.

a) $f(x) = 5 - x$ b) $f(x) = x^2 - 6x$

PRACTISING

4. a) Copy and complete the following table.

x	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$f(x)$	16	14	12	10	8	6	4	2	0	−2	−4	−6
$\dfrac{1}{f(x)}$												

b) Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

c) Find equations for $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

5. State the equation of the reciprocal of each function, and determine the equations of the vertical asymptotes of the reciprocal. Verify your results using graphing technology.

a) $f(x) = 2x$

b) $f(x) = x + 5$

c) $f(x) = x - 4$

d) $f(x) = 2x + 5$

e) $f(x) = -3x + 6$

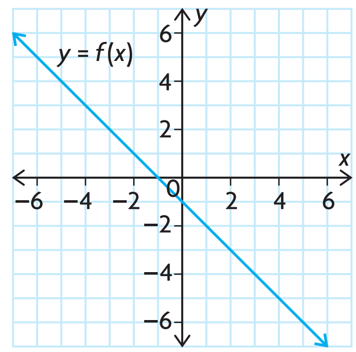
f) $f(x) = (x - 3)^2$

g) $f(x) = x^2 - 3x - 10$

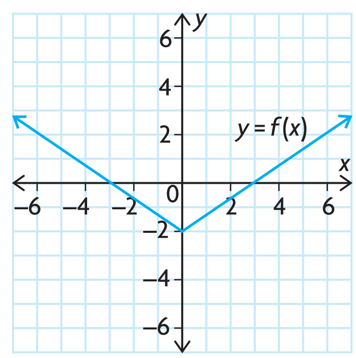
h) $f(x) = 3x^2 - 4x - 4$

6. Sketch the graph of the reciprocal of each function.

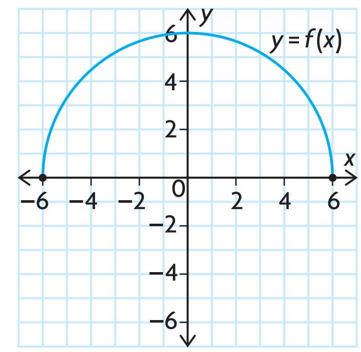
a)



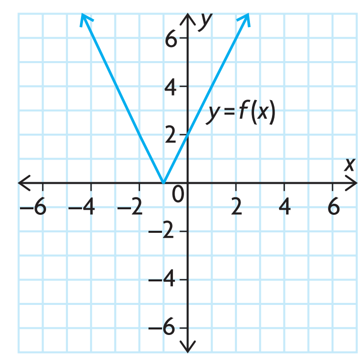
b)



c)



d)



7. Sketch each pair of graphs on the same axes. State the domain and range of each reciprocal function.

a) $y = 2x - 5, y = \dfrac{1}{2x - 5}$ **b)** $y = 3x + 4, y = \dfrac{1}{3x + 4}$

8. Draw the graph of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes.

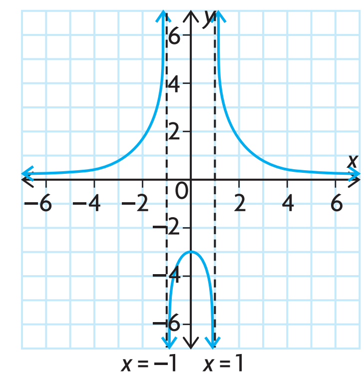
a) $f(x) = x^2 - 4$
b) $f(x) = (x - 2)^2 - 3$
c) $f(x) = x^2 - 3x + 2$
d) $f(x) = (x + 3)^2$
e) $f(x) = x^2 + 2$
f) $f(x) = -(x + 4)^2 + 1$

9. **K** For each function, determine the domain and range, intercepts, positive/negative intervals, and increasing/decreasing intervals. State the equation of the reciprocal function. Then sketch the graphs of the original and reciprocal functions on the same axes.

a) $f(x) = 2x + 8$
b) $f(x) = -4x - 3$
c) $f(x) = x^2 - x - 12$
d) $f(x) = -2x^2 + 10x - 12$

10. Why do the graphs of reciprocals of linear functions always have vertical asymptotes, but the graphs of reciprocals of quadratic functions sometimes do not? Provide sketches of three different reciprocal functions to illustrate your answer.

11. An equation of the form $y = \frac{k}{x^2 + bx + c}$ has a graph that closely matches the graph shown. Find the equation. Check your answer using graphing technology.



12. **A** A chemical company is testing the effectiveness of a new cleaning solution for killing bacteria. The test involves introducing the solution into a sample that contains approximately 10 000 bacteria. The number of bacteria remaining, $b(t)$, over time, t, in seconds is given by the equation $b(t) = 10\,000\frac{1}{t}$.

a) How many bacteria will be left after 20 s?
b) After how many seconds will only 5000 bacteria be left?
c) After how many seconds will only one bacterium be left?
d) This model is not always accurate. Determine what sort of inaccuracies this model might have. Assume that the solution was introduced at $t = 0$.
e) Based on these inaccuracies, what should the domain and range of the equation be?

13. **T** Use your graphing calculator to explore and then describe the key characteristics of the family of reciprocal functions of the form $g(x) = \frac{1}{x + n}$. Make sure that you include graphs to support your descriptions.

a) State the domain and range of $g(x)$.
b) For the family of functions $f(x) = x + n$, the y-intercept changes as the value of n changes. Describe how the y-intercept changes and how this affects $g(x)$.
c) If graphed, at what point would the two graphs $f(x)$ and $g(x)$ intersect?

14. **C** Due to a basketball tournament, your friend has missed this class. Write a concise explanation of the steps needed to graph a reciprocal function using the graph of the original function (without using graphing technology). Use an example, and explain the reason for each step.

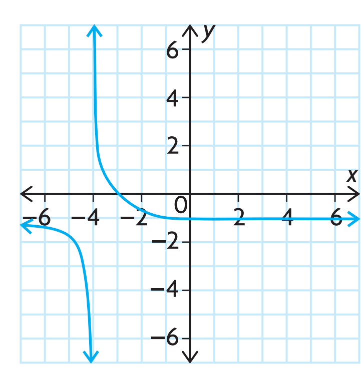
Extending

15. Sketch the graphs of the following reciprocal functions.

a) $y = \frac{1}{\sqrt{x}}$ **c)** $y = \frac{1}{2^x}$

b) $y = \frac{1}{x^3}$ **d)** $y = \frac{1}{\sin x}$

16. Determine the equation of the function in the graph shown.



5.2 Exploring Quotients of Polynomial Functions

YOU WILL NEED

- graph paper
- coloured pencils or pens
- graphing calculator or graphing software

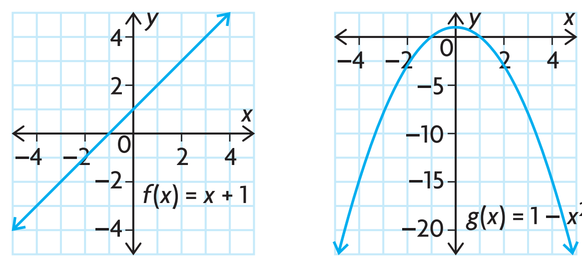
GOAL

Explore graphs that are created by dividing polynomial functions.

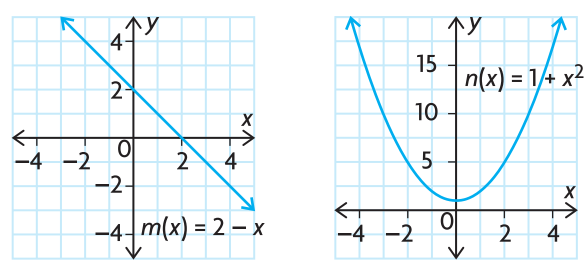
EXPLORE the Math

Each row shows the graphs of two polynomial functions.

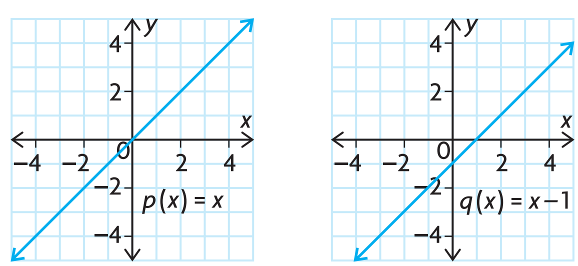
A.



B.



C.

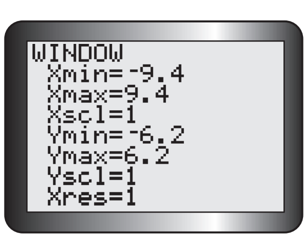


rational function

a function that can be expressed as $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, $q(x) \neq 0$ (e.g., $f(x) = \frac{3x^2 - 1}{x + 1}$, $x \neq -1$, and $f(x) = \frac{1 - x}{x^2}$, $x \neq 0$, are rational functions, but $f(x) = \frac{1 + x}{\sqrt{2 - x}}$, $x \neq 2$, is not because its denominator is not a polynomial)

? What are the characteristics of the graphs that are created by dividing two polynomial functions?

A. Using the given functions, write the equation of the rational function $y = \frac{f(x)}{g(x)}$. Enter this equation into Y1 of the equation editor of a graphing calculator. Graph this equation using the window settings shown, and draw a sketch.



B. Describe the characteristics of the graph you created in part A by answering the following questions:

i) Where are the zeros?
ii) Are there any asymptotes? If so, where are they?
iii) What are the domain and range of this function?
iv) Is it a **continuous function**? Explain.
v) Are there any values of $y = \frac{f(x)}{g(x)}$ that are undefined? What feature(s) of the graph is (are) related to these values?
vi) Describe the end behaviours of this function.
vii) Is the resulting graph a function? Explain.

C. Write the equation defined by $y = \frac{g(x)}{f(x)}$. Predict how the graph of this function will differ from the graph of $y = \frac{f(x)}{g(x)}$. Graph this function using your graphing calculator, and draw a sketch.

D. Describe the characteristics of the graph you created in part C by answering the questions in part B.

E. Repeat parts A through D for the functions in the other two rows.

F. Using graphing technology, and the same window settings you used in part A, explore the graphs of the following rational functions. Sketch each graph on separate axes, and note any holes or asymptotes.

i) $f(x) = \frac{x^2 - 1}{x - 1}$
ii) $f(x) = \frac{3}{x + 1}$
iii) $f(x) = \frac{x + 1}{x^2 - 2x - 3}$
iv) $f(x) = \frac{x + 1}{x + 2}$
v) $f(x) = \frac{0.5x^2 + 1}{x - 1}$
vi) $f(x) = \frac{x^2 + 2x}{x + 1}$
vii) $f(x) = \frac{9x}{1 + x^2}$
viii) $f(x) = \frac{2x^2 - 3}{x^2 + 1}$

G. Examine the graphs of the functions in parts i) and v) of part F at the point where $x = 1$. Explain why $f(x) = \frac{x^2 - 1}{x - 1}$ has a hole where $x = 1$, but $f(x) = \frac{0.5x^2 + 1}{x - 1}$ has a vertical asymptote. Identify the other functions in part F that have holes and the other functions that have vertical asymptotes.

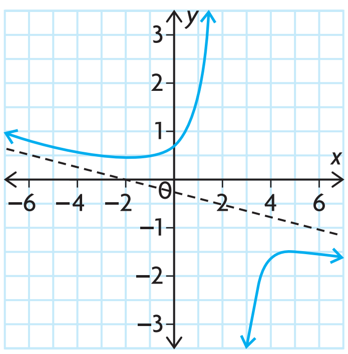
Tech *Support*

When entering a rational function into a graphing calculator, use brackets around the expression in the numerator and the expression in the denominator.

H. Redraw the graph of the rational function $f(x) = \frac{0.5x^2 + 1}{x - 1}$. Then enter the equation $y = 0.5x + 0.5$ into Y2 of the equation editor. What do you notice? Examine all your other sketches in this exploration to see if any of the other functions have an oblique asymptote.

oblique asymptote

an asymptote that is neither vertical nor horizontal, but slanted



I. Examine the equations with graphs that have horizontal asymptotes in part F. Compare the degree of the expression in the numerator with the degree of the expression in the denominator. Is there a connection between the degrees in the numerator and denominator and the existence of horizontal asymptotes? Explain. Repeat for functions with oblique asymptotes.

J. Investigate several functions of the form $f(x) = \frac{ax + b}{cx + d}$. Note similarities and differences. Without graphing, how can you predict where a horizontal asymptote will occur?

K. Investigate graphs of quotients of quadratic functions. How are they different from graphs of quotients of linear functions?

L. Summarize the different characteristics of the graphs of rational functions.

Reflecting

M. How do the zeros of the function in the numerator help you graph the rational function? How do the zeros of the function in the denominator help you graph the rational function?

N. Explain how you can use the expressions in the numerator and the denominator of a rational function to decide if the graph has
- **i)** a hole
- **ii)** a vertical asymptote
- **iii)** a horizontal asymptote
- **iv)** an oblique asymptote

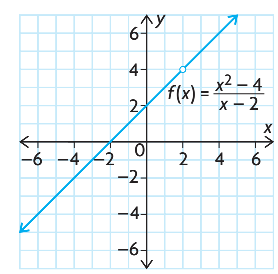
In Summary

Key Ideas

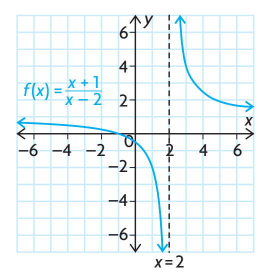
- The quotient of two polynomial functions results in a rational function which often has one or more discontinuities.
- The breaks or discontinuities in a rational function occur where the function is undefined. The function is undefined at values where the denominator is equal to zero. As a result, these values must be restricted from the domain of the function.
- The values that must be restricted from the domain of a rational function result in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).
- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

Need to Know

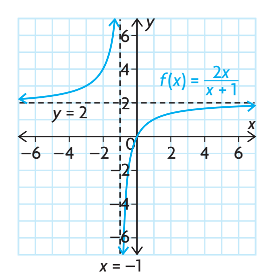
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a hole at $x = a$ if $\frac{p(a)}{q(a)} = \frac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x - a)$. For example, $f(x) = \frac{x^2 - 4}{x - 2}$ has the common factor of $(x - 2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x = 2$.



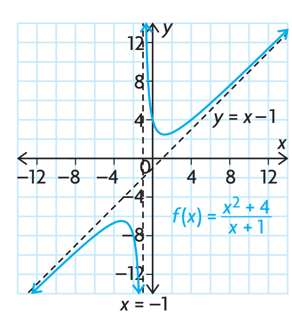
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a vertical asymptote at $x = a$ if $\frac{p(a)}{q(a)} = \frac{p(a)}{0}$. For example, $f(x) = \frac{x + 1}{x - 2}$ has a vertical asymptote at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x) = \frac{2x}{x + 1}$ has a horizontal asymptote at $y = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x) = \frac{x^2 + 4}{x + 1}$ has an oblique asymptote.



FURTHER Your Understanding

1. Without using graphing technology, match each equation with its corresponding graph. Explain your reasoning.

a) $y = \dfrac{-1}{x - 3}$

b) $y = \dfrac{x^2 - 9}{x - 3}$

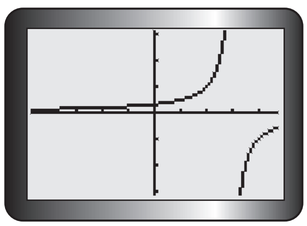
c) $y = \dfrac{1}{(x + 3)^2}$

d) $y = \dfrac{x}{(x - 1)(x + 3)}$

e) $y = \dfrac{1}{x^2 + 5}$

f) $y = \dfrac{x^2}{x - 3}$

A



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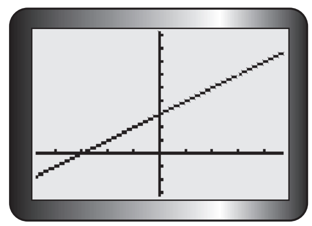

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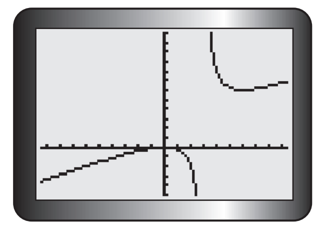

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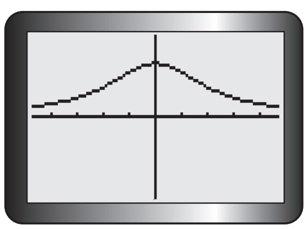

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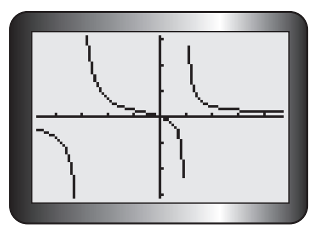
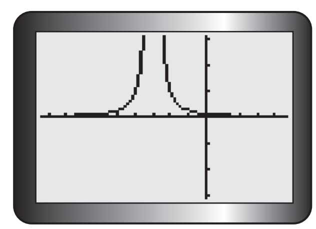


F



2. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal or oblique asymptotes.

a) $y = \dfrac{x}{x + 4}$

b) $y = \dfrac{1}{2x + 3}$

c) $y = \dfrac{2x + 5}{x - 6}$

d) $y = \dfrac{x^2 - 9}{x + 3}$

e) $y = \dfrac{1}{(x + 3)(x - 5)}$

f) $y = \dfrac{-x}{x + 1}$

g) $y = \dfrac{3x - 6}{x - 2}$

h) $y = \dfrac{-4x + 1}{2x - 5}$

i) $y = \dfrac{8x}{4x + 1}$

j) $y = \dfrac{x + 4}{x^2 - 16}$

k) $y = \dfrac{x}{5x - 3}$

l) $y = \dfrac{-3x + 1}{2x - 8}$

3. Write an equation for a rational function with the properties as given.

a) a hole at $x = 1$

b) a vertical asymptote anywhere and a horizontal asymptote along the x-axis

c) a hole at $x = -2$ and a vertical asymptote at $x = 1$

d) a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 2$

e) an oblique asymptote, but no vertical asymptote

5.3 Graphs of Rational Functions of the Form $f(x) = \frac{ax + b}{cx + d}$

GOAL

Sketch the graphs of rational functions, given equations of the form $f(x) = \frac{ax + b}{cx + d}$.

YOU WILL NEED

- graph paper
- graphing calculator or graphing software

INVESTIGATE the Math

The radius, in centimetres, of a circular juice blot on a piece of paper towel is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where t is measured in seconds. According to this model, the maximum size of the blot is determined by the location of the horizontal asymptote.

? How can you find the equation of the horizontal asymptote of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$?

A. Without graphing, determine the domain, intercepts, vertical asymptote, and positive/negative intervals of the simple rational function $f(x) = \frac{x}{x + 1}$.

B. Copy the following tables, and complete them by evaluating $f(x)$ for each value of x. Examine the **end behaviour** of $f(x)$ by observing the trend in $f(x)$ as x grows positively large and negatively large. What value does $f(x)$ seem to approach?

$x \to \infty$	
x	$f(x) = \frac{x}{x + 1}$
10	
100	
1 000	
10 000	
100 000	
1 000 000	

$x \to -\infty$	
x	$f(x) = \frac{x}{x + 1}$
−10	
−100	
−1 000	
−10 000	
−100 000	
−1 000 000	

C. Write an equation for the horizontal asymptote of the function in part B.

D. Repeat parts A, B, and C for the functions $g(x) = \frac{4x}{x + 1}$, $h(x) = \frac{2x}{3x + 1}$, and $m(x) = \frac{3x - 2}{2x - 5}$.

E. Verify your results by graphing all the functions in part D on a graphing calculator. Note similarities and differences among the graphs.

F. Make a list of the equations of the functions and the equations of their horizontal asymptotes. Discuss how the degree of the numerator compares with the degree of the denominator. Explain how the leading coefficients of x in the numerator and the denominator determine the equation of the horizontal asymptote.

G. Determine the equation of the horizontal asymptote of the juice blot function $r(t) = \frac{1 + 2t}{1 + t}$. What does this equation tell you about the eventual size of the juice blot?

Reflecting

H. How do the graphs of rational functions with linear expressions in the numerator and denominator compare with the graphs of reciprocal functions?

I. Explain how you determined the equation of a horizontal asymptote from

i) end behaviour tables

ii) the equation of the function

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine how a graph approaches a vertical asymptote

Determine how the graph of $f(x) = \frac{3x - 5}{x + 2}$ approaches its vertical asymptote.

Solution

$f(x) = \frac{3x - 5}{x + 2}$ has a vertical asymptote with the equation $x = -2$. Near this asymptote, the values of the function will grow very large in a positive direction or very large in a negative direction.

$f(x)$ is undefined when $x = -2$.

There is no common factor in the numerator and denominator.

Choose a value of x to the left and very close to -2. This value is less than -2.

$$f(-2.1) = \frac{3(-2.1) - 5}{(-2.1) + 2}$$
$$= 113$$

The graph of a rational function never crosses a vertical asymptote, so choose x-values that are very close to the vertical asymptote, on both sides, to determine the behaviour of the function.

On the left side of the vertical asymptote, the values of the function are positive. As $x \to -2$, $f(x) \to \infty$.

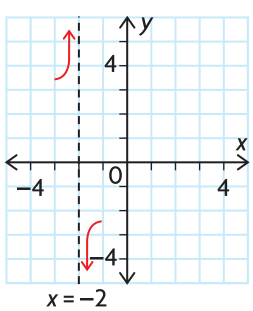
The function increases to large positive values as x approaches -2 from the left.

Choose a value of x to the right and very close to -2. This value is greater than -2.

$$f(-1.9) = \frac{3(-1.9) - 5}{(-1.9) + 2}$$
$$= -107$$

On the right side of the vertical asymptote, the values of the function are negative. As $x \to -2$, $f(x) \to -\infty$.

The function decreases to small negative values as x approaches -2 from the right.



Make a sketch to show how the graph approaches the vertical asymptote.

EXAMPLE 2 Using key characteristics to sketch the graph of a rational function

For each function,

a) $f(x) = \dfrac{2}{x - 3}$ **b)** $f(x) = \dfrac{x - 2}{3x + 4}$ **c)** $f(x) = \dfrac{x - 3}{2x - 6}$

i) determine the domain, intercepts, asymptotes, and positive/negative intervals
ii) use these characteristics to sketch the graph of the function
iii) describe where the function is increasing or decreasing

Solution

a) $f(x) = \dfrac{2}{x - 3}$

i) $D = \{x \in \mathbf{R} | x \neq 3\}$

The function $f(x) = \frac{2}{x - 3}$ is undefined when $x = 3$.

$f(0) = -\frac{2}{3}$, so the y-intercept is $-\frac{2}{3}$.

$f(x) \neq 0$, so there is no x-intercept.

Any rational function equals zero when its numerator equals zero. The numerator is always 2, so $f(x)$ can never equal zero.

The line $x = 3$ is a vertical asymptote.

The line $y = 0$ is a horizontal asymptote.

Since the numerator and denominator do not contain the common factor $(x - 3)$, $f(x)$ has a vertical asymptote at $x = 3$. Any rational function that is formed by a constant numerator and a linear function denominator has a horizontal asymptote at $y = 0$.

$f(x)$ is negative when $x \in (-\infty, 3)$ and positive when $x \in (3, \infty)$.

The numerator is always positive, so the denominator determines the sign of $f(x)$.

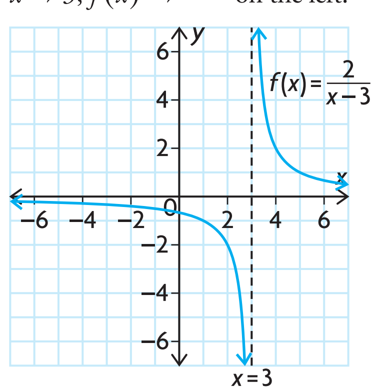
$x - 3 < 0$ when $x < 3$

$x - 3 > 0$ when $x > 3$

ii) Confirm the behaviour of $f(x)$ near the vertical asymptote.

$f(3.1) = 20$, so as $x \rightarrow 3, f(x) \rightarrow \infty$ on the right.

$f(2.9) = -20$, so as $x \rightarrow 3, f(x) \rightarrow -\infty$ on the left.



Use all the information in part i) to sketch the graph.

iii) From the graph, the function is decreasing on its entire domain: when $x \in (-\infty, 3)$ and when $x \in (3, \infty)$.

b) $f(x) = \dfrac{x - 2}{3x + 4}$

i) $3x + 4 \neq 0$
$3x \neq -4$
$x \neq -\dfrac{4}{3}$

$D = \left\{x \in \mathbf{R} \mid x \neq -\dfrac{4}{3}\right\}$

$f(x)$ is undefined when the denominator is zero.

$f(0) = \dfrac{0 - 2}{3(0) + 4} = \dfrac{-2}{4}$ or $-\dfrac{1}{2}$

To determine the y-intercept, let $x = 0$.

The y-intercept is $-\dfrac{1}{2}$.

$f(x) = 0$ when $\dfrac{x - 2}{3x + 4} = 0.$
$x - 2 = 0$
$x = 2$

The x-intercept is 2.

To determine the x-intercept, let $y = 0$. Any rational function equals zero when its numerator equals zero.

The line $x = -\frac{4}{3}$ is a vertical asymptote.

This is the value that makes $f(x)$ undefined.

The line $y = \frac{1}{3}$ is a horizontal asymptote.

The ratio of the leading coefficients of the numerator and denominator is $\frac{1}{3}$.

Examine the signs of the numerator and denominator, and their quotient, to determine the positive/negative intervals.

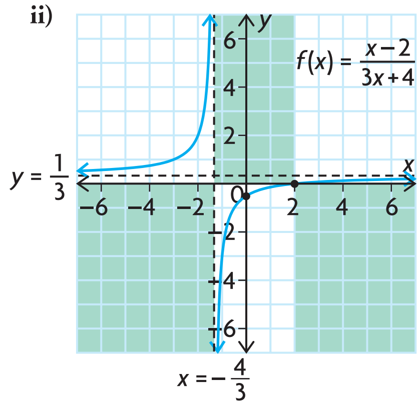
	$x < -\frac{4}{3}$	$-\frac{4}{3} < x < 2$	$x > 2$
$x - 2$	$-$	$-$	$+$
$3x + 4$	$-$	$+$	$+$
$\frac{x - 2}{3x - 4}$	$\frac{-}{-} = +$	$\frac{-}{+} = -$	$\frac{+}{+} = +$

The vertical asymptote and the x-intercept divide the set of real numbers into three intervals: $\left(-\infty, -\frac{4}{3}\right)$, $\left(-\frac{4}{3}, 2\right)$, and $(2, \infty)$. Choose numbers in each interval to evaluate the sign of each expression.

$f(x)$ is positive when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in (2, \infty)$.

$f(x)$ is negative when $x \in \left(-\frac{4}{3}, 2\right)$.

ii)



When sketching the graph, it helps to shade the regions where there is no graph. Use the positive and negative intervals as indicators for these regions. For example, since $f(x)$ is positive on $\left(-\infty, -\frac{4}{3}\right)$, there is no graph under the x-axis on this interval. Draw the asymptotes, and mark the intercepts. Then draw the graph to approach the asymptotes.

iii) From the graph, $f(x)$ is increasing on its entire domain; that is, when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in \left(-\frac{4}{3}, \infty\right)$.

c) $f(x) = \dfrac{x-3}{2x-6}$

i)
$$2x - 6 \neq 0$$
$$2x \neq 6$$
$$x \neq 3$$

$f(x)$ is undefined when the denominator is zero.

$D = \{x \in \mathbf{R} | x \neq 3\}$

$$f(0) = \frac{0-3}{2(0)-6} = \frac{-3}{-6} \text{ or } \frac{1}{2}$$

To determine the y-intercept, let $x = 0$.

The y-intercept is $\frac{1}{2}$.

$f(x) \neq 0$, so there is no x-intercept.

To determine the x-intercept, let $y = 0$. Only consider when the numerator is zero; that is, when $x - 3 = 0$. Therefore, the numerator is zero at $x = 3$, but this has already been excluded from the domain.

$$f(x) = \frac{x-3}{2(x-3)}$$

$f(x)$ has a hole, not a vertical asymptote, where $x = 3$.

Factoring reveals a common factor $(x - 3)$ in the numerator and denominator. The graph has a hole at the point where $x = 3$.

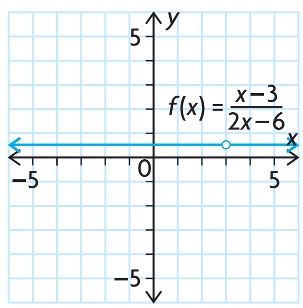
$$f(x) = \frac{\overset{1}{\cancel{x-3}}}{2(\underset{1}{\cancel{x-3}})} = \frac{1}{2}$$

The value of the function is always $\frac{1}{2}$ for all values of x, except when $x = 3$.

$f(x)$ is positive at all points in its domain.

$f(x)$ has no vertical asymptote or x-intercept. There is only one interval to consider: $(-\infty, \infty)$. For any value of x, $f(x) = \frac{1}{2}$.

ii) The graph is a horizontal line with the equation $y = \frac{1}{2}$. There is a hole at $x = 3$.



Use the information in part i) to sketch the graph.

iii) The function is neither increasing nor decreasing. It is constant on its entire domain.

EXAMPLE 3 Solving a problem by graphing a rational function

The function $P(t) = \dfrac{30(7t + 9)}{3t + 2}$ models the population, in thousands, of a town t years since 1990. Describe the population of the town over the next 20 years.

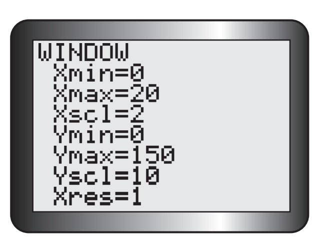
Solution

$$P(t) = \frac{30(7t + 9)}{3t + 2}$$

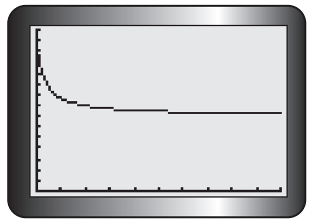
Use the equation to help you decide on the window settings. For the given context, $t \geq 0$ and $P(t) > 0$.

Determine the initial population in 1990, when $t = 0$.

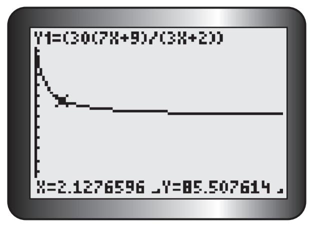
$$P(0) = \frac{30(7(0) + 9)}{3(0) + 2} = 135$$



Graph $P(t)$ to show the population for the 20 years after 1990.



The value that makes $3t + 2 = 0$ lies outside the domain of $P(t)$. There is no vertical asymptote in the domain.

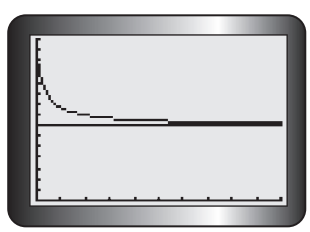


TRACE along the curve to get an idea of how the population changed.

In the first two years, the population dropped by about 50 000 people. Then it began to level off and approach a steady value.

There is a horizontal asymptote at $P = \frac{30(7)}{3} = 70$.

Use the function equation to determine the equation of the horizontal asymptote. For large values of t, $P(t) \doteq \frac{30(7t)}{3t}$. Therefore, the leading coefficients in the numerator and denominator define the equation of the horizontal asymptote.



The population of the town has been decreasing since 1990. It was 135 000 in 1990, but dropped by about 50 000 in the next two years. Since then, the population has begun to level off and, according to the model, will approach a steady value of 70 000 people by 2010.

Multiply the values of the function by 1000, since the population is given in thousands.

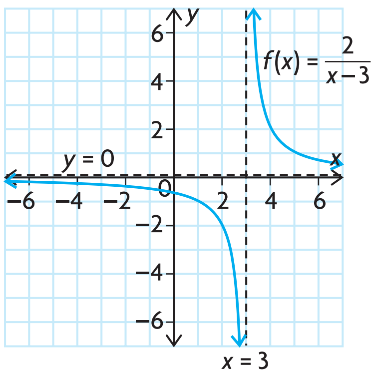
In Summary

Key Ideas

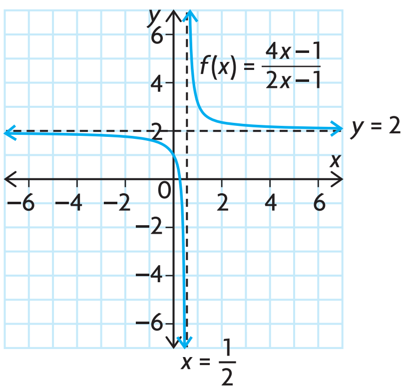
- The graphs of most rational functions of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$ have both a vertical asymptote and a horizontal asymptote.
- You can determine the equation of the vertical asymptote directly from the equation of the function by finding the zero of the denominator.
- You can determine the equation of the horizontal asymptote directly from the equation of the function by examining the ratio of the leading coefficients in the numerator and the denominator. This gives you the end behaviours of the function.
- To sketch the graph of a rational function, you can use the domain, intercepts, equations of asymptotes, and positive/negative intervals.

Need to Know

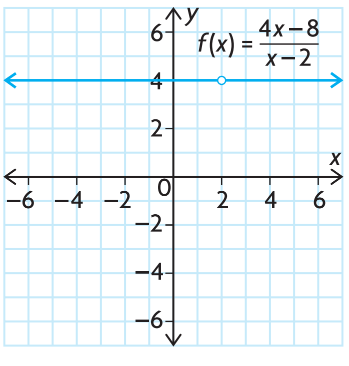
- Rational functions of the form $f(x) = \frac{b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = 0$. For example, see the graph of $f(x) = \frac{2}{x - 3}$.



- Most rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = \frac{a}{c}$. For example, see the graph of $f(x) = \frac{4x - 1}{2x - 1}$.



The exception occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factor occurs. As a result, the graph has no asymptotes. For example, see the graph of $f(x) = \frac{4x - 8}{x - 2} = \frac{4(x - 2)}{(x - 2)}$.



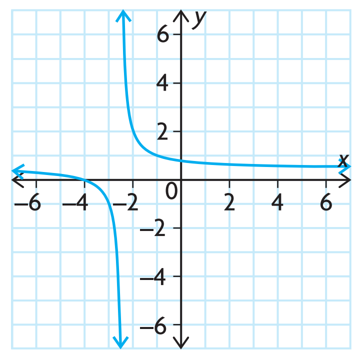
CHECK *Your Understanding*

1. Match each function with its graph.

a) $h(x) = \dfrac{x + 4}{2x + 5}$ **c)** $f(x) = \dfrac{3}{x - 1}$

b) $m(x) = \dfrac{2x - 4}{x - 2}$ **d)** $g(x) = \dfrac{2x - 3}{x + 2}$

A



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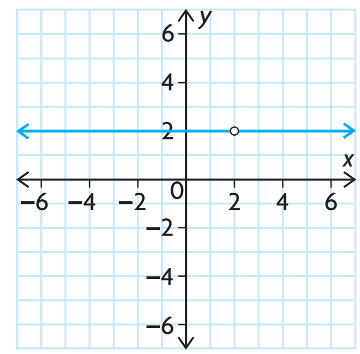

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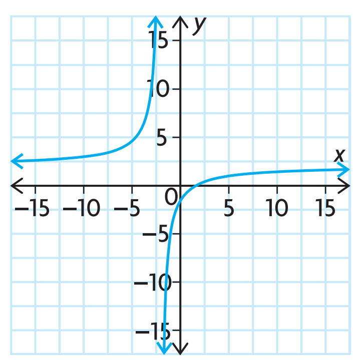

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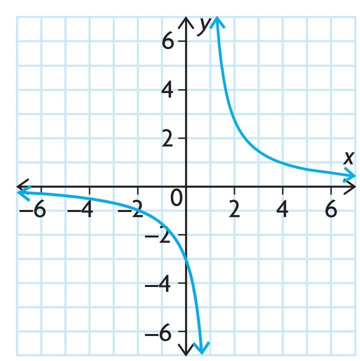


2. Consider the function $f(x) = \frac{3}{x - 2}$.

a) State the equation of the vertical asymptote.

b) Use a table of values to determine the behaviour(s) of the function near its vertical asymptote.

c) State the equation of the horizontal asymptote.

d) Use a table of values to determine the end behaviours of the function near its horizontal asymptote.

e) Determine the domain and range.

f) Determine the positive and negative intervals.

g) Sketch the graph.

3. Repeat question 2 for the rational function $f(x) = \frac{4x - 3}{x + 1}$.

PRACTISING

4. State the equation of the vertical asymptote of each function. Then choose a strategy to determine how the graph of the function approaches its vertical asymptote.

a) $y = \frac{2}{x + 3}$ **c)** $y = \frac{2x + 1}{2x - 1}$

b) $y = \frac{x - 1}{x - 5}$ **d)** $y = \frac{3x + 9}{4x + 1}$

5. K For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \frac{3}{x + 5}$ **c)** $f(x) = \frac{x + 5}{4x - 1}$

b) $f(x) = \frac{10}{2x - 5}$ **d)** $f(x) = \frac{x + 2}{5(x + 2)}$

6. Read each set of conditions. State the equation of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$ that meets these conditions, and sketch the graph.

a) vertical asymptote at $x = -2$, horizontal asymptote at $y = 0$; negative when $x \in (-\infty, -2)$, positive when $x \in (-2, \infty)$; always decreasing

b) vertical asymptote at $x = -2$, horizontal asymptote at $y = 1$; x-intercept $= 0$, y-intercept $= 0$; positive when $x \in (-\infty, -2)$ or $(0, \infty)$, negative when $x \in (-2, 0)$

c) hole at $x = 3$; no vertical asymptotes; y-intercept $= (0, 0.5)$

d) vertical asymptotes at $x = -2$ and $x = 6$, horizontal asymptote at $y = 0$; positive when $x \in (-\infty, -2)$ or $(6, \infty)$, negative when $x \in (-2, 6)$; increasing when $x \in (-\infty, 2)$, decreasing when $x \in (2, \infty)$

7. T **a)** Use a graphing calculator to investigate the similarities and differences in the graphs of rational functions of the form $f(x) = \frac{8x}{nx + 1}$, for $n = 1, 2, 4$, and 8.

b) Use your answer for part a) to make a conjecture about how the function changes as the values of n approach infinity.

c) If n is negative, how does the function change as the value of n approaches negative infinity? Choose your own values, and use them as examples to support your conclusions.

8. Without using a graphing calculator, compare the graphs of the rational functions $f(x) = \frac{3x + 4}{x - 1}$ and $g(x) = \frac{x - 1}{2x + 3}$.

9. The function $I(t) = \frac{15t + 25}{t}$ gives the value of an investment, in thousands of dollars, over t years.

a) What is the value of the investment after 2 years?
b) What is the value of the investment after 1 year?
c) What is the value of the investment after 6 months?
d) There is an asymptote on the graph of the function at $t = 0$. Does this make sense? Explain why or why not.
e) Choose a very small value of t (a value near zero). Calculate the value of the investment at this time. Do you think that the function is accurate at this time? Why or why not?
f) As time passes, what will the value of the investment approach?

10. A An amount of chlorine is added to a swimming pool that contains pure water. The concentration of chlorine, c, in the pool at t hours is given by $c(t) = \frac{2t}{2 + t}$, where c is measured in milligrams per litre. What happens to the concentration of chlorine in the pool during the 24 h period after the chlorine is added?

11. C Describe the key characteristics of the graphs of rational functions of the form $f(x) = \frac{ax + b}{cx + d}$. Explain how you can determine these characteristics using the equations of the functions. In what ways are the graphs of all the functions in this family alike? In what ways are they different? Use examples in your comparison.

Extending

12. Not all asymptotes are horizontal or vertical. Find a rational function that has an asymptote that is neither horizontal nor vertical, but slanted or oblique.

13. Use long division to rewrite $f(x) = \frac{2x^3 - 7x^2 + 8x - 5}{x - 1}$ in the form $f(x) = ax^2 + bx + c + \frac{k}{x - 1}$. What does this tell you about the end behaviour of the function? Graph the function. Include all asymptotes in your graph. Write the equations of the asymptotes.

14. Let $f(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $g(x) = \frac{x^3 + 8}{x^2 + 9}$, $h(x) = \frac{x^3 - 3x}{x + 1}$, and $m(x) = \frac{x^2 + x - 12}{x^2 - 4}$.

a) Which of these rational functions has a horizontal asymptote?
b) Which has an oblique asymptote?
c) Which has no vertical asymptote?
d) Graph $y = m(x)$, showing the asymptotes and intercepts.

5 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you use the graph of a linear or quadratic function to graph its reciprocal function?

A: If you have the graph of a linear or quadratic function, you can draw the graph of its reciprocal function as follows:

1. Draw a vertical asymptote for the reciprocal function at each zero of the original function. The x-axis is a horizontal asymptote for the reciprocal function, unless the original function is a constant function.
2. The reciprocal function has the same positive/negative intervals that the original function has, so you can shade the regions where there will be no graph.
3. Mark any points where the y-value of the original function is 1 or -1. The reciprocal function also goes through these points.
4. The y-intercept of the reciprocal function is the reciprocal of the y-intercept of the original function. Determine and mark the y-intercept of the reciprocal function.
5. If the original function is quadratic, the reciprocal function has a local maximum/minimum at the same x-value as the vertex of the quadratic function. The y-value of this local maximum/minimum is the reciprocal of the y-value of the vertex. Determine and mark the local maximum/minimum point.
6. Draw the pieces of the graph of the reciprocal function through the marked points, approaching the asymptotes.

Study *Aid*

- See Lesson 5.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1 and 2.

Q: What are rational functions, and what are the characteristics of their graphs?

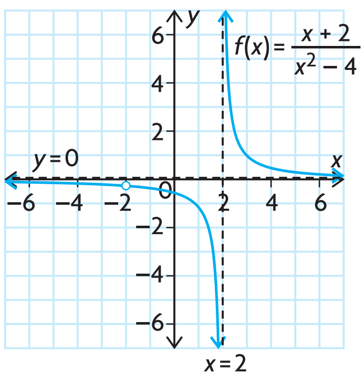
A: Rational functions are quotients of polynomial functions. Their equations have the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Graphs of rational functions may have vertical, horizontal, or oblique asymptotes. Some rational functions have holes in their graphs.

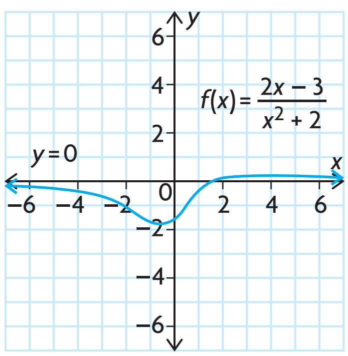
Study *Aid*

- See Lesson 5.2.
- Try Mid-Chapter Review Question 3.

Study Aid

- See Lesson 5.2.
- Try Mid-Chapter Review Question 4.





Study Aid

- See Lesson 5.3, Example 2.
- Try Mid-Chapter Review Questions 5 to 8.

Q: Most rational functions have one or more discontinuities. Where and why do these discontinuities occur? When is a rational function continuous?

A: If the polynomial function in the denominator of a rational function has one or more zeros, the rational function will be discontinuous at these points. If a value of x can be zero in both the numerator and the denominator of a rational function (that is, if the numerator and denominator have a common linear factor), the result is a hole. This type of discontinuity is called a point discontinuity.

If a zero in the denominator does not correspond to a zero in the numerator, there will be a vertical asymptote at the x-value. This is called an infinite discontinuity.

For example, $f(x) = \frac{x + 2}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)}$ has a point discontinuity where $x = -2$ because -2 is a zero of both the denominator, $q(x) = x^2 - 4$, and the numerator, $p(x) = x + 2$. The graph of $f(x)$ has an infinite discontinuity where $x = 2$ because 2 is a zero of $q(x)$ but not of $p(x)$. The graph also has a hole at $x = -2$ and a vertical asymptote at $x = 2$. Note that $\frac{p(-2)}{q(-2)} = \frac{0}{0}$, but $\frac{p(2)}{q(2)} = \frac{4}{0}$.

If the polynomial function in the denominator of a rational function does not have any zeros, the rational function is continuous. Its graph is a smooth curve, with no breaks.

For example, $f(x) = \frac{2x - 3}{x^2 + 2}$ is a continuous rational function with a horizontal asymptote at $y = 0$.

Q: How do you determine the equations of the vertical and horizontal asymptotes of a rational function of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$?

A: You can determine the equations of the vertical and horizontal asymptotes directly from the equation of a rational function of the form $f(x) = \frac{b}{cx + d}$ or $f(x) = \frac{ax + b}{cx + d}$. The vertical asymptote occurs at the zero of the function in the denominator. The equation of the vertical asymptote is $x = -\frac{d}{c}$. The horizontal asymptote describes the end behaviour of the function when $x \to \pm\infty$.

All rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have $y = \frac{a}{c}$ as a horizontal asymptote.

All rational functions of the form $f(x) = \frac{b}{cx + d}$ have $y = 0$ (the x-axis) as a horizontal asymptote.

PRACTICE Questions

Lesson 5.1

1. State the reciprocal of each function, and determine the locations of any vertical asymptotes.

a) $f(x) = x - 3$
b) $f(q) = -4q + 6$
c) $f(z) = z^2 + 4z - 5$
d) $f(d) = 6d^2 + 7d - 3$

2. For each function, determine the domain and range, intercepts, positive/negative intervals, and intervals of increase/decrease. Use this information to sketch the graphs of the function and its reciprocal.

a) $f(x) = 4x + 6$
b) $f(x) = x^2 - 4$
c) $f(x) = x^2 + 6$
d) $f(x) = -2x - 4$

Lesson 5.2

3. Different characteristics of the graph of a rational function are created by different characteristics of the function. List at least four characteristics of a graph and the characteristic of the function that causes each one. Make sure that you include a characteristic of a continuous rational function in your list.

4. For each function, determine the equations of any vertical asymptotes, the locations of any holes, and the existence of any horizontal asymptotes (other than the x-axis) or oblique asymptotes.

a) $y = \dfrac{x}{x - 2}$
b) $y = \dfrac{x - 1}{3x - 3}$
c) $y = \dfrac{-7x}{4x + 2}$
d) $y = \dfrac{x^2 + 2}{x - 6}$
e) $y = \dfrac{1}{x^2 + 2x - 15}$

Lesson 5.3

5. List the functions that had a horizontal asymptote in question 4, and give the equation of the horizontal asymptote.

6. For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \dfrac{5}{x - 6}$
b) $f(x) = \dfrac{3x}{x + 4}$
c) $f(x) = \dfrac{5x + 10}{x + 2}$
d) $f(x) = \dfrac{x - 2}{2x - 1}$

7. Kevin is trying to develop a reciprocal function to model some data that he has. He wants the horizontal asymptote to be $y = 7$. He also wants the graph to decrease and approach $y = 7$ as x approaches infinity, so he chooses the equation $y = \frac{7x + 6}{x}$. Then he decides that he needs the vertical asymptote to be $x = -1$, so he changes the equation to $y = \frac{7x + 6}{x + 1}$. What happened to the graph of Kevin's function? Did it give him the result he wanted? Explain why or why not.

8. For the function $f(x) = \frac{7x - m}{2 - nx}$, find the values of m and n such that the vertical asymptote is at $x = 6$ and the x-intercept is 5.

9. Create a rational function that has a domain of $\{x \in \mathbf{R} | x \neq -2\}$ and no vertical asymptote. Describe the graph of this function.

5.4 Solving Rational Equations

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Connect the solution to a rational equation with the graph of a rational function.

LEARN ABOUT the Math

When they work together, Stuart and Lucy can deliver flyers to all the homes in their neighbourhood in 42 min. When Lucy works alone, she can finish the deliveries in 13 min less time than Stuart can when he works alone.

? When Stuart works alone, how long does he take to deliver the flyers?

EXAMPLE 1 Selecting a strategy to solve a rational equation

Determine the time that Stuart takes to deliver the flyers when he works alone.

Solution A: Creating an equation and solving it using algebra

Let s minutes be the time that Stuart takes to deliver the flyers when working alone.

Lucy takes $(s - 13)$ minutes when working alone.

Choose a variable to represent Stuart's time and use it to write an expression for Lucy's time.

Lucy delivers the flyers in 13 min less time than Stuart.

The fraction of deliveries made in one minute

- by Stuart working alone is $\frac{1}{s}$
- by Lucy working alone is $\frac{1}{s - 13}$
- by Stuart and Lucy working together is $\frac{1}{42}$

$$\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$$

Compare the rates at which they work.
For example, if Stuart took 80 min to deliver all the flyers, he would deliver $\frac{1}{80}$ of the flyers per minute.
$s > 13$ because Stuart takes longer than Lucy to deliver the flyers, and it is not possible for the denominators to be zero.

Multiply by the LCD.

$$42s(s-13)\left(\frac{1}{s}+\frac{1}{s-13}\right)=42s(s-13)\left(\frac{1}{42}\right)$$

$$\frac{42s(s-13)}{s}+\frac{42s(s-13)}{s-13}=\frac{42s(s-13)}{42}$$

$$\frac{42\cancel{s}^{1}(s-13)}{\cancel{s}_{1}}+\frac{42s\cancel{(s-13)}^{1}}{\cancel{s-13}_{1}}=\frac{\cancel{42}^{1}s(s-13)}{\cancel{42}_{1}}$$

$$42(s-13)+42s=s(s-13)$$

There are no common factors in the denominators, so the LCD (lowest common denominator) is the product of the three denominators $42s(s-13)$. Multiply each term by the LCD, and then simplify the resulting rational expressions to remove all the denominators.

$$\begin{aligned}42s-546+42s&=s^2-13s\\0&=s^2-97s+546\\0&=(s-91)(s-6)\\s&=6\text{ or }91\end{aligned}$$

Solve the quadratic equation by factoring or by using the quadratic formula.

$s > 13$ so 6 is not an admissible solution.

Remember to look for inadmissible solutions by carefully considering both the context and the information given in the problem.

$$\begin{aligned}\text{LS}&=\frac{1}{s}+\frac{1}{s-13}\\&=\frac{1}{91}+\frac{1}{91-13}\\&=\frac{1}{91}+\frac{1}{78}\\&=\frac{78+91}{7098}\\&=\frac{169}{7098}\\&=\frac{1}{42}\end{aligned}$$

Check the solution, $s = 91$, by substituting it into the original equation.

$$\text{RS}=\frac{1}{42}$$

Since LS = RS, $s = 91$ is the solution.

It will take Stuart 91 min to deliver the flyers when working alone.

Solution B: Using the graph of a rational function to solve a rational equation

The equation that models the problem is $\frac{1}{s} + \frac{1}{s - 13} = \frac{1}{42}$, where s represents the time, in minutes, that Stuart takes to deliver the flyers when working alone.

$$\frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42} = 0$$

Subtract $\frac{1}{42}$ from each side.

To solve the equation, find the zeros of the function $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

Graph $f(s) = \frac{1}{s} + \frac{1}{s - 13} - \frac{1}{42}$.

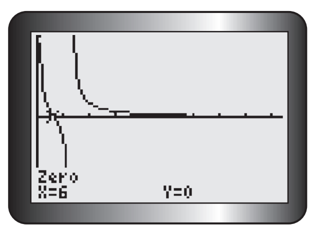
Use the zero operation to determine the zeros.

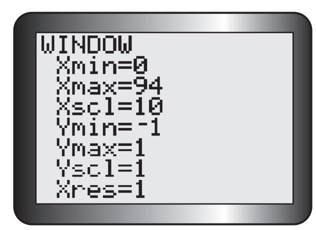
From the equation, you can expect the graph to have vertical asymptotes at $s = 0$ and $s = 13$.

Since you are only interested in finding the zeros, you can limit the y-values to those close to zero.

Tech Support

For help determining the zeros on a graphing calculator, see Technical Appendix, T-8.



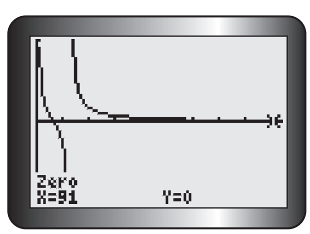


The first zero for $f(s)$ is $s = 6$. Reject this solution since $s > 13$.

Determine the other zero.

You know that Lucy takes 13 min less time than Stuart takes, so Stuart must take longer than 13 min.

The solution $s = 6$ is inadmissible.



The solution is $s = 91$.

Stuart takes 91 min to deliver the flyers when working alone.

Reflecting

A. In Solution A, explain how a rational equation was created using the times given in the problem.

B. In Solution B, explain how finding the zeros of a rational function provided the solution to the problem.

C. Where did the inadmissible root obtained in Solution A show up in the graphical solution in Solution B? How was this root dealt with?

APPLY *the Math*

EXAMPLE 2 Using an algebraic strategy to solve simple rational equations

Solve each rational equation.

a) $\dfrac{x-2}{x-3} = 0$ **b)** $\dfrac{x+3}{x-4} = \dfrac{x-1}{x+2}$

Solution

a) $\dfrac{x-2}{x-3} = 0,\ x \neq 3$

Determine any restrictions on the value of x.

$$\cancel{(x-3)}^{1}\frac{(x-2)}{\cancel{(x-3)}_{1}} = 0(x-3)$$

$$x - 2 = 0$$

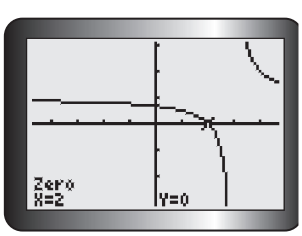
$$x = 2$$

Multiply both sides of the equation by the LCD, $(x-3)$.

Add 2 to each side.

To verify, graph $f(x) = \dfrac{x-2}{x-3}$ and use the zero operation to determine the zero.

From the equation, the graph will have a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 1$.



The solution is $x = 2$.

b) $\dfrac{x+3}{x-4} = \dfrac{x-1}{x+2},\ x \neq -2, 4$

Note the restrictions.

$$(x-4)(x+2)\left(\frac{x+3}{x-4}\right) = (x-4)(x+2)\left(\frac{x-1}{x+2}\right)$$

Multiply each side of the equation by the LCD, $(x-4)(x+2)$.

$$\overset{1}{\cancel{(x-4)}}(x+2)\left(\frac{x+3}{\underset{1}{\cancel{x-4}}}\right) = (x-4)\overset{1}{\cancel{(x+2)}}\left(\frac{x-1}{\underset{1}{\cancel{x+2}}}\right)$$

Simplify.

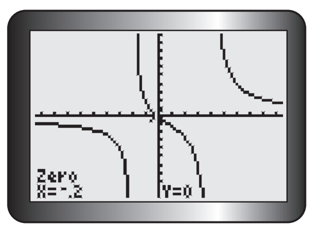
$$(x+2)(x+3) = (x-4)(x-1)$$
$$x^2 + 5x + 6 = x^2 - 5x + 4$$

Expand. Subtract x^2, and add $5x$ to both sides.

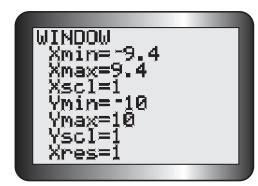
$$10x + 6 = 4$$
$$10x = -2$$
$$x = -0.2$$

Solve the resulting linear equation.

To verify, graph $f(x) = \dfrac{x+3}{x-4} - \dfrac{x-1}{x+2}$ and use the zero operation to determine the zero.



Adjust the window settings so you can view enough of the graph to see all the possible zeros.



The solution is $x = -0.2$.

EXAMPLE 3 Connecting the solution to a problem with the zeros of a rational function

Salt water is flowing into a large tank that contains pure water. The concentration of salt, c, in the tank at t minutes is given by $c(t) = \dfrac{10t}{25+t}$, where c is measured in grams per litre. When does the salt concentration in the tank reach 3.75 g/L?

Solution

If the salt concentration is 3.75, $c(t) = 3.75$.

$$\frac{10t}{25+t} = 3.75$$

$$(25+t)\left(\frac{10t}{25+t}\right) = 3.75(25+t)$$

$$\overset{1}{\cancel{(25+t)}}\frac{10t}{\underset{1}{\cancel{(25+t)}}} = 3.75(25+t)$$

$$10t = 93.75 + 3.75t$$

Set the function expression equal to 3.75. $25 + t \neq 0$, and because t measures the time since the salt water started flowing, $t \geq 0$.

Multiply both sides of the equation by the LCD, $(25 + t)$, and solve the resulting linear equation.

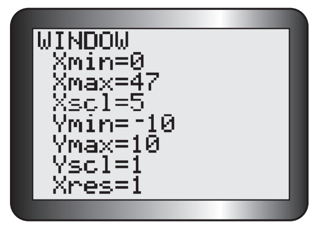
$$10t - 3.75t = 93.75$$
$$6.25t = 93.75$$
$$\frac{6.25t}{6.25} = \frac{93.75}{6.25}$$
$$t = 15$$

Use inverse operations to solve for t.

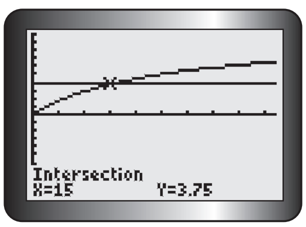
It takes 15 min for the salt concentration to reach 3.75 g/L.

To verify, graph $f(t) = \frac{10t}{25 + t}$ and $g(t) = 3.75$, and determine where the functions intersect.

Use an appropriate window setting, based on the domain, $t \geq 0$.



Use the intersect operation.



The salt concentration reaches 3.75 g/L after 15 min.

Tech Support

For help determining the point of intersection between two functions, see Technical Appendix, T-12.

EXAMPLE 4 Using a rational function to model and solve a problem

Rima bought a case of concert T-shirts for \$450. She kept two T-shirts for herself and sold the rest for \$560, making a profit of \$10 on each T-shirt. How many T-shirts were in the case?

Solution

Let the number of T-shirts in the case be x.

Buying price per T-shirt $= \frac{450}{x}$

Selling price per T-shirt $= \frac{560}{x - 2}$

Rima paid \$450 for x T-shirts, so each T-shirt cost her $\$\frac{450}{x}$.

She kept two for herself, which left $x - 2$ T-shirts for her to sell.

Rima sold $x - 2$ T-shirts for \$560, so she charged $\$\frac{560}{x - 2}$ for each one.

$$\frac{560}{x-2} - \frac{450}{x} = 10$$

She made a profit of \$10 on each T-shirt, so the difference between the selling price and the buying price was \$10.

$$x(x-2)\left(\frac{560}{x-2} - \frac{450}{x}\right) = 10x(x-2)$$

Multiply both sides of the equation by the LCD, $x(x - 2)$.

$$\frac{560x\cancel{(x-2)}^{1}}{\cancel{x-2}_{1}} - \frac{450\cancel{x}^{1}(x-2)}{\cancel{x}_{1}} = 10x(x-2)$$

$$560x - 450(x-2) = 10x(x-2)$$
$$560x - 450x + 900 = 10x^2 - 20x$$
$$0 = 10x^2 - 130x - 900$$

Expand and collect all terms to one side of the equation.

$$0 = 10(x^2 - 13x - 90)$$
$$0 = 10(x-18)(x+5)$$
$$x = 18 \text{ or } -5$$

Solve the resulting quadratic equation by factoring.

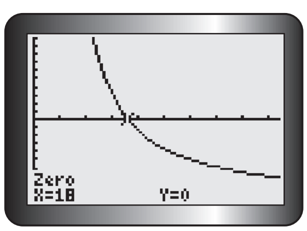
-5 is inadmissible since $x \geq 0$.
There were 18 T-shirts in the case.

You cannot have a negative number of T-shirts in the case.

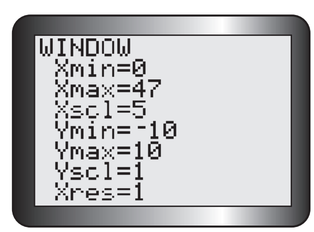
To verify, graph $f(x) = \frac{560}{x-2} - \frac{450}{x} - 10$ and determine the zeros using the zero operation.

If $\frac{560}{x-2} - \frac{450}{x} = 10$, then $\frac{560}{x-2} - \frac{450}{x} - 10 = 0$.

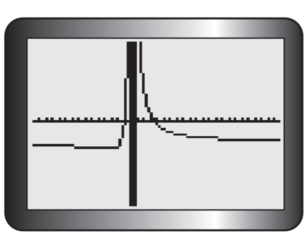
Zeros for $f(x)$ are possible solutions to the problem.



Use an appropriate window setting, based on the domain, $x \geq 0$.



The zero occurs when $x = 18$.
Zoom out to check that there are no other zeros in the domain.



The other zero is for a negative value of x, which is inadmissible in the context of this problem.

There is no other zero in the domain.
There were 18 T-shirts in the case.

In Summary

Key Ideas

- You can solve a rational equation algebraically by multiplying each term in the equation by the lowest common denominator and solving the resulting polynomial equation.
- The root of the equation $\frac{ax + b}{cx + d} = 0$ is the zero (x-intercept) of the function $f(x) = \frac{ax + b}{cx + d}$.
- You can use graphing technology to solve a rational equation or verify the solution. Determine the zeros of the corresponding rational function, or determine the intersection of two functions.

Need to Know

- The zeros of a rational function are the zeros of the function in the numerator.
- Reciprocal functions do not have zeros. All functions of the form $f(x) = \frac{1}{g(x)}$ have the x-axis as a horizontal asymptote. They do not intersect the x-axis.
- When solving contextual problems, it is important to check for inadmissible solutions that are outside the domain determined by the context.
- When using a graphing calculator to determine a zero or intersection point, you can avoid inadmissible roots by matching the window settings to the domain of the function in the context of the problem.

CHECK Your Understanding

1. Are $x = 3$ and $x = -2$ solutions to the equation $\frac{2}{x} = \frac{x - 1}{3}$? Explain how you know.

2. Solve each equation algebraically. Then verify your solution using graphing technology.

a) $\frac{x + 3}{x - 1} = 0$

b) $\frac{x + 3}{x - 1} = 2$

c) $\frac{x + 3}{x - 1} = 2x + 1$

d) $\frac{3}{3x + 2} = \frac{6}{5x}$

3. For each rational equation, write a function whose zeros are the solutions.

a) $\frac{x - 3}{x + 3} = 2$

b) $\frac{3x - 1}{x} = \frac{5}{2}$

c) $\frac{x - 1}{x} = \frac{x + 1}{x + 3}$

d) $\frac{x - 2}{x + 3} = \frac{x - 4}{x + 5}$

4. Solve each equation in question 3 algebraically, and verify your solution using a graphing calculator.

PRACTISING

5. Solve each equation algebraically.

a) $\frac{2}{x} + \frac{5}{3} = \frac{7}{x}$

b) $\frac{10}{x + 3} + \frac{10}{3} = 6$

c) $\frac{2x}{x - 3} = 1 - \frac{6}{x - 3}$

d) $\frac{2}{x + 1} + \frac{1}{x + 1} = 3$

e) $\frac{2}{2x + 1} = \frac{5}{4 - x}$

f) $\frac{5}{x - 2} = \frac{4}{x + 3}$

6. Solve each equation algebraically.

a) $\frac{2x}{2x + 1} = \frac{5}{4 - x}$

b) $\frac{3}{x} + \frac{4}{x + 1} = 2$

c) $\frac{2x}{5} = \frac{x^2 - 5x}{5x}$

d) $x + \frac{x}{x - 2} = 0$

e) $\frac{1}{x + 2} + \frac{24}{x + 3} = 13$

f) $\frac{-2}{x - 1} = \frac{x - 8}{x + 1}$

7. Solve each equation using graphing technology. Round your answers to two decimal places, if necessary.

a) $\frac{2}{x + 2} = \frac{3}{x + 6}$

b) $\frac{2x - 5}{x + 10} = \frac{1}{x - 6}$

c) $\frac{1}{x - 3} = \frac{x + 2}{7x + 14}$

d) $\frac{1}{x} - \frac{1}{45} = \frac{1}{2x - 3}$

e) $\frac{2x + 3}{3x - 1} = \frac{x + 2}{4}$

f) $\frac{1}{x} = \frac{2}{x} + 1 + \frac{1}{1 - x}$

8. **K** a) Use algebra to solve $\frac{x + 1}{x - 2} = \frac{x + 3}{x - 4}$. Explain your steps.

b) Verify your answer in part a) using substitution.

c) Verify your answer in part a) using a graphing calculator.

9. The Greek mathematician Pythagoras is credited with the discovery of the Golden Rectangle. This is considered to be the rectangle with the dimensions that are the most visually appealing. In a Golden Rectangle, the length and width are related by the proportion $\frac{l}{w} = \frac{w}{l - w}$. A billboard with a length of 15 m is going to be built. What must its width be to form a Golden Rectangle?

10. The Turtledove Chocolate factory has two chocolate machines. Machine A takes s minutes to fill a case with chocolates, and machine B takes $s + 10$ minutes to fill a case. Working together, the two machines take 15 min to fill a case. Approximately how long does each machine take to fill a case?

11. Tayla purchased a large box of comic books for \$300. She gave 15 of the comic books to her brother and then sold the rest on an Internet website for \$330, making a profit of \$1.50 on each one. How many comic books were in the box? What was the original price of each comic book?

12. A Polluted water flows into a pond. The concentration of pollutant, c, in the pond at time t minutes is modelled by the equation $c(t) = 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)$, where c is measured in kilograms per cubic metre.

a) When will the concentration of pollutant in the pond reach $6\ \text{kg/m}^3$?

b) What will happen to the concentration of pollutant over time?

13. T Three employees work at a shipping warehouse. Tom can fill an order in s minutes. Paco can fill an order in $s - 2$ minutes. Carl can fill an order in $s + 1$ minutes. When Tom and Paco work together, they take about 1 minute and 20 seconds to fill an order. When Paco and Carl work together, they take about 1 minute and 30 seconds to fill an order.

a) How long does each person take to fill an order?

b) How long would all three of them, working together, take to fill an order?

14. C Compare and contrast the different methods you can use to solve a rational equation. Make a list of the advantages and disadvantages of each method.

Extending

15. Solve $\frac{x^2 - 6x + 5}{x^2 - 2x - 3} = \frac{2 - 3x}{x^2 + 3x + 3}$ correct to two decimal places.

16. Objects A and B move along a straight line. Their positions, s, with respect to an origin, at t seconds, are modelled by the following functions:

Object A: $s(t) = \frac{7t}{t^2 + 1}$

Object B: $s(t) = t + \frac{5}{t + 2}$

a) When are the objects at the same position?

b) When is object A closer to the origin than object B?

5.5 Solving Rational Inequalities

YOU WILL NEED

- graphing calculator

GOAL

Solve rational inequalities using algebraic and graphical approaches.

LEARN ABOUT the Math

The function $P(t) = \frac{20t}{t + 1}$ models the population, in thousands, of Nickelford, t years after 1997. The population, in thousands, of nearby New Ironfield is modelled by $Q(t) = \frac{240}{t + 8}$.

rational inequality

a statement that one rational expression is less than or greater than another rational expression $\left(\text{e.g., } \frac{2x}{x + 3} > \frac{x - 1}{5x}\right)$

? How can you determine the time period when the population of New Ironfield exceeded the population of Nickelford?

EXAMPLE 1 Selecting a strategy to solve a problem

Determine the interval(s) of t where the values of $Q(t)$ are greater than the values of $P(t)$.

Solution A: Using an algebraic strategy to solve an inequality

$$\frac{240}{t + 8} > \frac{20t}{t + 1}$$

The population of New Ironfield exceeds the population of Nickelford when $Q(t) > P(t)$. $t \geq 0$ in the context of this problem. There are no other restrictions on the expressions in the rational inequality since the values that make both expressions undefined are negative numbers.

$$(t+8)(t+1)\left(\frac{240}{t+8}\right) > (t+8)(t+1)\left(\frac{20t}{t+1}\right)$$

$$\overset{1}{\cancel{(t+8)}}(t+1)\left(\frac{240}{\underset{1}{\cancel{t+8}}}\right) > (t+8)\overset{1}{\cancel{(t+1)}}\left(\frac{20t}{\underset{1}{\cancel{t+1}}}\right)$$

Multiply both sides of the inequality by the LCD. The value of the LCD is always positive, since $t \geq 0$, so the inequality sign is unchanged.

$$\begin{aligned}
240(t+1) &> 20t(t+8)\\
240t + 240 &> 20t^2 + 160t\\
0 &> 20t^2 + 160t - 240t - 240\\
0 &> 20t^2 - 80t - 240\\
0 &> 20(t^2 - 4t - 12)\\
0 &> 20(t-6)(t+2)
\end{aligned}$$

Expand and simplify both sides. Then subtract $240t$ and 240 from both sides.

Factor the resulting quadratic expression.

Examine the sign of the factored polynomial expression on the right side of the inequality.

	$t < -2$	$-2 < t < 6$	$t > 6$
$20(t-6)$	$-$	$-$	$+$
$t+2$	$-$	$+$	$+$
$20(t-6)(t+2)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$

The inequality $0 > 20(t-6)(t+2)$ is true when the expression on the right side is negative. The sign of the factored quadratic expression changes when $t = -2$ and when $t = 6$, because the expression is zero at these values. Use a table to determine when the sign of the expression is negative on each side of these values.

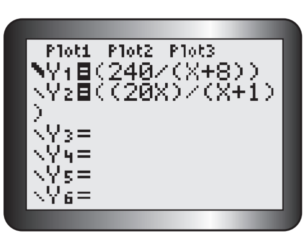
The inequality $0 > 20(t-6)(t+2)$ is true when $-2 < t < 6$.

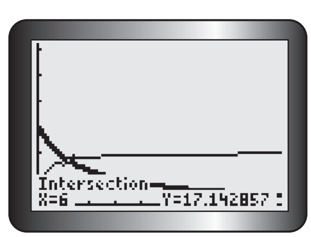
The population of New Ironfield exceeded the population of Nickelford for six years after 1997, until 2003.

Since the domain is $t \geq 0$, however, numbers that are negative cannot be included. Therefore, the solution is $0 \leq t < 6$.

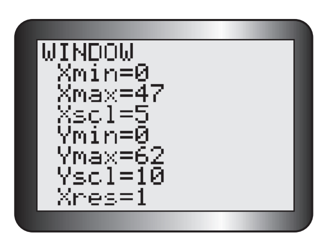
Solution B: Solving a rational inequality by graphing two rational functions

To solve $Q(t) > P(t)$, graph $Q(t) = \dfrac{240}{t+8}$ and $P(t) = \dfrac{20t}{t+1}$ using graphing technology, and determine the value of t at the intersection point(s).





It helps to bold the graph of $Q(t)$ so you can remember which graph is which. Use window settings that reflect the domain of the functions.



There is only one intersection within the domain of the functions.

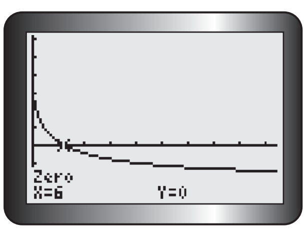
From the graphs, $Q(t) > P(t)$ for $0 \leq t < 6$.
The population of New Ironfield exceeded the population of Nickelford until 2003.

If $Q(t) > P(t)$, the graph of $Q(t)$ lies above the graph of $P(t)$. Looking at the graphs, this is true for the parts of the graph of $Q(t)$ up to the intersection point at $t = 6$. The graphs will not intersect again because each graph is approaching a different horizontal asymptote. From the defining equations, the graph of $Q(t)$ is approaching the line $Q = 0$ while the graph of $P(t)$ is approaching the line $P = 20$.

Solution C: Solving a rational inequality by determining the zeros of a combined function

When $Q(t) > P(t)$, $Q(t) - P(t) > 0$.

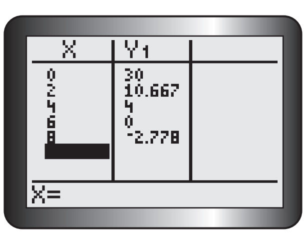
Graph $f(t) = Q(t) - P(t) = \dfrac{240}{t + 8} - \dfrac{20t}{t + 1}$ and use the zero operation to locate the zero.



Combine the two population functions into a single function, $f(t) = Q(t) - P(t)$. When $Q(t) > P(t)$, $f(t)$ will have positive values.

When a function has positive values, its graph lies above the x-axis.

The graph is above the x-axis for $0 \leq t < 6$.



By examining the values of $f(t)$ in a table, you can verify that the function continues to decrease but remains positive when $0 \leq t < 6$.

$f(t)$ has positive values for $0 \leq t < 6$.
For the six years after 1997, the population of New Ironfield exceeded the population of Nickelford.

Reflecting

A. How is the solution to an inequality different from the solution to an equation?

B. In Solution A, how was the rational inequality manipulated to obtain a simpler quadratic inequality?

C. In Solution B, how were the graphs of the related rational functions used to find the solution to an inequality?

D. In Solution C, how did creating a new function help to solve the inequality?

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an inequality that involves a linear function and a reciprocal function

Solve $x - 2 < \frac{8}{x}$.

Solution A: Using an algebraic strategy and a sign chart

$$x - 2 < \frac{8}{x}, x \neq 0$$

$$x - 2 - \frac{8}{x} < 0$$

$$\frac{x^2}{x} - \frac{2x}{x} - \frac{8}{x} < 0$$

$$\frac{x^2 - 2x - 8}{x} < 0$$

$$\frac{(x - 4)(x + 2)}{x} < 0$$

Determine any restrictions on x. Subtract $\frac{8}{x}$ from both sides.

x is the LCD and it can be positive or negative. Multiplying both sides by x would require that two cases be considered, since the inequality sign must be reversed when multiplying by a negative. The alternative is to create an expression with a common denominator, x.

Combine the terms to create a single rational expression.

Factor the numerator.

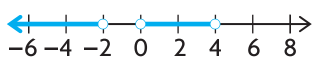
Examine the sign of the rational expression.

	$x < -2$	$-2 < x < 0$	$0 < x < 4$	$x > 4$
$x - 4$	$-$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$\dfrac{(x-4)(x+2)}{x}$	$\dfrac{(-)(-)}{-} = -$	$\dfrac{(-)(+)}{-} = +$	$\dfrac{(-)(+)}{+} = -$	$\dfrac{(+)(+)}{+} = +$

The sign of a rational expression changes each time the sign of one of its factors changes. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is negative.

The overall expression is negative when $x < -2$ or when $0 < x < 4$.

The inequality is true when $x \in (-\infty, -2)$ or $x \in (0, 4)$.



Write the solution in interval or set notation, and draw the solution set on a number line.

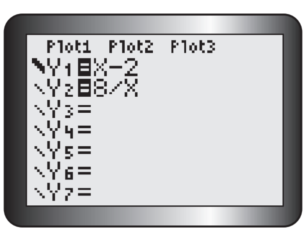
Solution B: Using graphing technology

$x - 2 < \dfrac{8}{x}, x \neq 0$

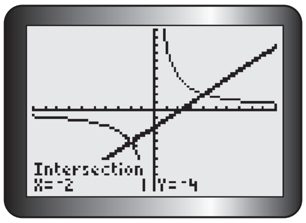
Let $f(x) = x - 2$ and $g(x) = \dfrac{8}{x}$.

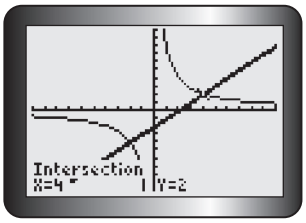
The solution set for the inequality will be all x-values for which $f(x) < g(x)$.

Write each side of the inequality as its own function. Enter both functions in the equation editor, using a bold line for $f(x)$.



Graph $f(x)$ and $g(x)$ on the same axes, and use the intersect operation to determine the intersection points.





$f(x) < g(x)$ where the bold graph of $f(x)$ lies beneath the graph of $g(x)$. Notice that the bold linear function is above the reciprocal function on the left side and close to the vertical asymptote, $x = 0$. It is below the reciprocal function on the right side and close to this asymptote.

$f(x) < g(x)$ when $x < -2$ or when $0 < x < 4$.

The solution set is $\{x \in \mathbf{R} | x < -2 \text{ or } 0 < x < 4\}$.

You can also use interval notation or a number line to describe the solution set, as in Solution A.

EXAMPLE 3 Determining the solution set for an inequality that involves two rational functions

Determine the solution set for the inequality $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$.

Solution A: Using algebra and a sign chart

Rewrite $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}, x \neq -1, 3,$

as $\frac{x+3}{x+1} - \frac{x-2}{x-3} \geq 0.$

Note the restrictions on x.

Subtract $\frac{x-2}{x-3}$ from both sides to create an inequality with zero on the right side.

$$\frac{(x-3)(x+3)}{(x-3)(x+1)} - \frac{(x-2)(x+1)}{(x-3)(x+1)} \geq 0$$

$$\frac{x^2 - 9 - (x^2 - x - 2)}{(x-3)(x+1)} \geq 0$$

Subtract the rational expressions on the left side using a common denominator.

$$\frac{x^2 - 9 - x^2 + x + 2}{(x-3)(x+1)} \geq 0$$

Expand and simplify the numerator.

$$\frac{x-7}{(x-3)(x+1)} \geq 0$$

A rational expression is zero when its numerator is zero.

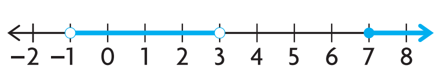
The rational expression is equal to zero when $x = 7$, so 7 is included in the solution set.

Examine the sign of the simplified rational expression on the intervals shown to determine where the rational expression is greater than zero.

	$x < -1$	$-1 < x < 3$	$3 < x < 7$	$x > 7$
$x - 7$	$-$	$-$	$-$	$+$
$x - 3$	$-$	$-$	$+$	$+$
$x + 1$	$-$	$+$	$+$	$+$
$\frac{(x-7)}{(x-3)(x+1)}$	$\frac{-}{(-)(-)} = -$	$\frac{-}{(-)(+)} = +$	$\frac{-}{(+)(+)} = -$	$\frac{+}{(+)(+)} = +$

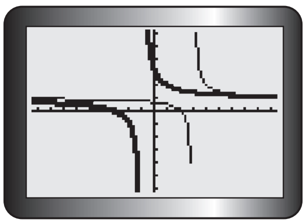
The expression is undefined at $x = -1$ and $x = 3$. It is equal to 0 at $x = 7$. These numbers create four intervals to consider. Choose a test value in each interval to determine the sign of each part of the expression. Then determine the intervals where the overall expression is positive.

The solution set is $\{x \in \mathbf{R} \mid -1 < x < 3 \text{ or } x \geq 7\}$.



Solution B: Using graphing technology

$$\frac{x+3}{x+1} \geq \frac{x-2}{x-3}, x \neq -1, 3$$

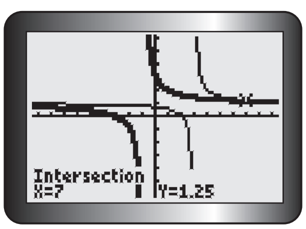


Use each side of the inequality to define a function. Graph $f(x) = \frac{x+3}{x+1}$ with a bold line and $g(x) = \frac{x-2}{x-3}$ with a regular line.

The graph of $f(x)$ has a vertical asymptote at $x = -1$.
The graph for $g(x)$ has a vertical asymptote at $x = 3$.
Both graphs have $y = 1$ as a horizontal asymptote.

Determine the equations of the asymptotes from the equations of the functions.

Use the intersect operation to locate any intersection points.



It looks as though the graphs might intersect on the left side of the screen, as well as on the right side. No matter how far you trace along the left branches, however, you never reach a point where the *y*-value is the same on both curves.

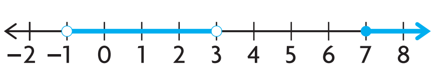
The functions are equal when $x = 7$.

$f(x) > g(x)$ between the asymptotes at $x = -1$ and $x = 3$, and for $x > 7$.

$f(x) = g(x)$ when $x = 7$.

The bold graph of $f(x)$ is above the graph of $g(x)$ between the two vertical asymptotes and then after the intersection point.

The solution set for $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$ is



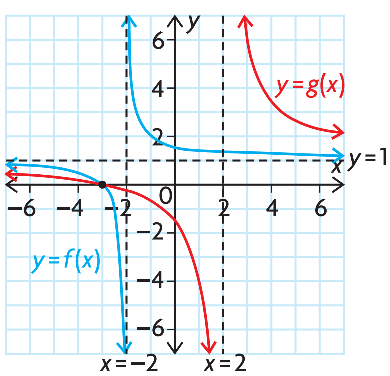
In Summary

Key Ideas

- Solving an inequality means finding all the possible values of the variable that satisfy the inequality.
- To solve a rational inequality algebraically, rearrange the inequality so that one side is zero. Combine the expressions on the no-zero side using a common denominator. Make a table to examine the sign of each factor and the sign of the entire expression on the intervals created by the zeros of the numerator and the denominator.
- Only when you are certain that each denominator is positive can you multiply both sides by the lowest common denominator to make the inequality easier to solve.
- You can always solve a rational inequality using graphing technology.

Need to Know

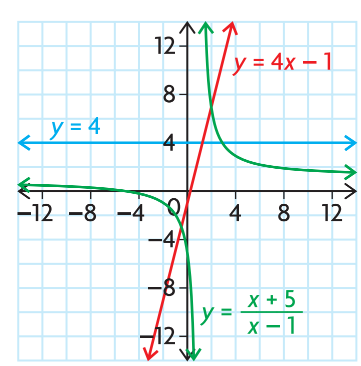
- When multiplying or dividing both sides of an inequality by a negative it is necessary to reverse the inequality sign to maintain equivalence.
- You can solve an inequality using graphing technology by graphing the functions on each side of the inequality sign and then identifying all the intervals created by the vertical asymptotes and points of intersection. For x-values that satisfy $f(x) > g(x)$, identify the specific intervals where the graph of $f(x)$ is above the graph of $g(x)$. For x-values that satisfy $f(x) < g(x)$, identify the specific intervals where the graph of $f(x)$ is below the graph of $g(x)$.
  Consider the following graph:



In this graph, there are four intervals to consider: $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$ and $(2, \infty)$. In these intervals, $f(x) > g(x)$ when $x \in (-\infty, -3)$ or $(-2, 2)$, and $f(x) < g(x)$ when $x \in (-3, -2)$ or $(2, \infty)$.

- You can also solve an inequality using graphing technology by creating an equivalent inequality with zero on one side and then identifying the intervals created by the zeros on the graph of the new function. Finding where the graph lies above the x-axis (where $f(x) > 0$) or below the x-axis (where $f(x) < 0$) defines the solutions to the inequality.

CHECK Your Understanding



1. Use the graph shown to determine the solution set for each of the following inequalities.

a) $\dfrac{x+5}{x-1} < 4$ **b)** $4x - 1 > \dfrac{x+5}{x-1}$

2. **a)** Show that the inequality $\dfrac{6x}{x+3} \le 4$ is equivalent to the inequality $\dfrac{2(x-6)}{(x+3)} \le 0$.

b) Sketch the solution on a number line.

c) Write the solution using interval notation.

3. a) Show that the inequality $x + 2 > \frac{15}{x}$ is equivalent to the inequality $\frac{(x + 5)(x - 3)}{x} > 0$.

b) Use a table to determine the positive/negative intervals for $f(x) = \frac{(x + 5)(x - 3)}{x}$.

c) State the solution to the inequality using both set notation and interval notation.

PRACTISING

4. Use algebra to find the solution set for each inequality. Verify your answer using graphing technology.

a) $\frac{1}{x + 5} > 2$

b) $\frac{1}{2x + 10} < \frac{1}{x + 3}$

c) $\frac{3}{x - 2} < \frac{4}{x}$

d) $\frac{7}{x - 3} \geq \frac{2}{x + 4}$

e) $\frac{-6}{x + 1} > \frac{1}{x}$

f) $\frac{-5}{x - 4} < \frac{3}{x + 1}$

5. Use algebra to obtain a factorable expression from each inequality, if necessary. Then use a table to determine interval(s) in which the inequality is true.

a) $\frac{t^2 - t - 12}{t - 1} < 0$

b) $\frac{t^2 + t - 6}{t - 4} \geq 0$

c) $\frac{6t^2 - 5t + 1}{2t + 1} > 0$

d) $t - 1 < \frac{30}{5t}$

e) $\frac{2t - 10}{t} > t + 5$

f) $\frac{-t}{4t - 1} \geq \frac{2}{t - 9}$

6. Use graphing technology to solve each inequality.

a) $\frac{x + 3}{x - 4} \geq \frac{x - 1}{x + 6}$

b) $x + 5 < \frac{x}{2x + 6}$

c) $\frac{x}{x + 4} \leq \frac{1}{x + 1}$

d) $\frac{x}{x + 9} \geq \frac{1}{x + 1}$

e) $\frac{x - 8}{x} > 3 - x$

f) $\frac{x^2 - 16}{(x - 1)^2} \geq 0$

7. a) Find all the values of x that make the following inequality true:
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$\frac{3x - 8}{2x - 1} > \frac{x - 4}{x + 1}$

b) Graph the solution set on a number line. Write the solution set using interval notation and set notation.

8. **a)** Use an algebraic strategy to solve the inequality $\frac{-6t}{t-2} < \frac{-30}{t-2}$.
b) Graph both inequalities to verify your solution.
c) Can these rational expressions be used to model a real-world situation? Explain.

9. **A** The equation $f(t) = \frac{5t}{t^2 + 3t + 2}$ models the bacteria count, in thousands, for a sample of tap water that is left to sit over time, t, in days. The equation $g(t) = \frac{15t}{t^2 + 9}$ models the bacteria count, in thousands, for a sample of pond water that is also left to sit over several days. In both models, $t > 0$. Will the bacteria count for the tap water sample ever exceed the bacteria count for the pond water? Justify your answer.

10. Consider the inequality $0.5x - 2 < \frac{5}{2x}$.
a) Rewrite the inequality so that there is a single, simplified expression on one side and a zero on the other side.
b) List all the factors of the rational expression in a table, and determine on which intervals the inequality is true.

11. **T** An economist for a sporting goods company estimates the revenue and cost functions for the production of a new snowboard. These functions are $R(x) = -x^2 + 10x$ and $C(x) = 4x + 5$, respectively, where x is the number of snowboards produced, in thousands. The average profit is defined by the function $AP(x) = \frac{P(x)}{x}$, where $P(x)$ is the profit function. Determine the production levels that make $AP(x) > 0$.

12. **C** **a)** Explain why the inequalities $\frac{x+1}{x-1} < \frac{x+3}{x+2}$ and $\frac{x+5}{(x-1)(x+2)} < 0$ are equivalent.
b) Describe how you would use a graphing calculator to solve these inequalities.
c) Explain how you would use a table to solve these inequalities.

Extending

13. Solve $\left|\frac{x}{x-4}\right| \geq 1$.

14. Solve $\frac{1}{\sin x} < 4, 0° \leq x \leq 360°$.

15. Solve $\frac{\cos(x)}{x} > 0.5, 0° < x < 90°$.